

THE SMALL QUANTUM COHOMOLOGY OF A WEIGHTED PROJECTIVE SPACE, A MIRROR D -MODULE AND THEIR CLASSICAL LIMITS

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ABSTRACT. We first describe a canonical mirror partner (B -model) of the small quantum orbifold cohomology of weighted projective spaces (A -model) in the framework of differential equations: we attach to the A -model (*resp.* B -model) a D -module on the torus and we show that these two D -modules are isomorphic. This makes the A and B -models mirror partners and yields, in this situation, an explicit and finer version of a recent result of Iritani. Then we study, using the theory of the Kashiwara-Malgrange filtration, their degenerations at the origin and we apply our results to the construction of (classical, limit, logarithmic) Frobenius manifolds.

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1. INTRODUCTION

Mirror symmetry has different mathematical formulations: equality between the I and J functions, equivalence of categories, isomorphisms of Frobenius manifolds *etc...* In this paper, we first explore the D -module aspect for the weighted projective spaces $\mathbb{P}(w) := \mathbb{P}(w_0, w_1, \dots, w_n)$, the A -model, where w_0, w_1, \dots, w_n are positive integers (to simplify the exposition, we will assume that $w_0 = 1$). It will be encoded by the *Saito structures of weight n* on $\mathbb{P}^1 \times M$, that is tuples (M, H, ∇, S, n) where M is a complex manifold, H is a trivial bundle on $\mathbb{P}^1 \times M$, ∇ is a flat meromorphic connection with logarithmic poles at $\{\infty\} \times M$ and of order 1 at $\{0\} \times M$ and S is a symmetric, nondegenerate, ∇ -flat bilinear form (for short a *metric*, even if there is no positivity consideration here). More precisely, we attach a Saito structure on $\mathbb{P}^1 \times \mathbb{C}^*$ to the small quantum orbifold cohomology of $\mathbb{P}(w)$ and we show that it is isomorphic to the one associated with a suitable potential: this B -model will be our mirror partner for the small quantum orbifold cohomology of weighted projective spaces. It should be emphasized that our construction yields an explicit and finer version of [20, Proposition 4.8] (Iritani's B -models are *a priori* only bundles on $\mathbb{C} \times \mathbb{C}^*$), using a more direct, algebraic, approach and enables us to understand the results of [8] using the theory of differential systems.

In order to get this first result, we proceed as follows: following Iritani [20], we first attach a Saito structure to any proper smooth Deligne-Mumford stack using the quantum orbifold cohomology. Thanks to the results recently obtained in [8], this construction can be done very explicitly in the case of the weighted projective spaces and yields, taking into account an action of the Picard group, a Saito structure

$$(\mathcal{M}_A, \tilde{H}^{A, \text{sm}}, \tilde{\nabla}^{A, \text{sm}}, \tilde{S}^{A, \text{sm}}, n)$$

where $\mathcal{M}_A = H^2(\mathbb{P}(w), \mathbb{C}) / \text{Pic}(\mathbb{P}(w)) \simeq \mathbb{C}^*$, the metric $\tilde{S}^{A, \text{sm}}$ being constructed with the help of the orbifold Poincaré duality. We will call this Saito structure the (*small*) A -model D -module. It should be noticed, and this will be a crucial observation, that the usual sections $1_{f_i} P^j$ of the orbifold cohomology are not *global* sections of the bundle $\tilde{H}^{A, \text{sm}}$.

We then look for a mirror partner of this A -model D -module. Using the methods developed in [14] and [22], we show how it is canonically obtained from the Gauss-Manin system of the function $F : U \times \mathcal{M}_B \rightarrow \mathbb{C}$ defined by

$$F(u_1, \dots, u_n, x) = u_1 + \dots + u_n + \frac{x}{u_1^{w_1} \dots u_n^{w_n}}$$

where $U = (\mathbb{C}^*)^n$ and $\mathcal{M}_B = \mathbb{C}^*$. Indeed, a canonical solution of the Birkhoff problem for the Brieskorn lattice of F gives a canonical trivial bundle H^B on $\mathbb{P}^1 \times \mathcal{M}_B$ equipped with a connection with the desired poles (and it should be emphasized that we have something canonical here). We get in this way a canonical Saito structure

$$(\mathcal{M}_B, H^B, \nabla^B, S^B, n)$$

which will be our B -model D -module, and we show that the A -model D -module and the B -model D -module are isomorphic (see Theorem 5.1.1).

Identifying the A -model D -module and the B -model D -module, we obtain finally a canonical Saito structure

$$\mathcal{S}_w = (\mathcal{M}, H, \nabla, S, n)$$

where $\mathcal{M} = \mathbb{C}^*$ (the index w recalls the weights w_0, \dots, w_n) and, as a by-product, a canonical Frobenius type structure \mathbb{F}_w on \mathcal{M} in the sense of [11] and [18], that is a tuple

$$\mathbb{F}_w = (\mathcal{M}, E, R_0, R_\infty, \Phi, \nabla, g)$$

the different objects involved satisfying some natural compatibility relations (coming from the flatness of ∇). This Frobenius type structure will be the main tool in our construction of Frobenius manifolds.

In the second part of this paper, we study the behaviour of these structures at the origin (this kind of problem is also considered in [10], using another strategy and in a different situation). We construct (see section 6.1.2) a canonical limit Saito structure

$$\overline{\mathcal{S}}_w = (\overline{H}, \overline{\nabla}, \overline{S}, n)$$

on \mathbb{P}^1 , using Deligne's extensions of the connection involved, and thus a canonical limit Frobenius type structure $\overline{\mathbb{F}}_w$ on a point. We explain how it can be used to understand the correspondance between "classical limits", that is between the orbifold cohomology ring of $\mathbb{P}(w)$ and a suitable graded vector space: we hope that it will throw new light on [22, theorem 1.1].

The last part is devoted to the construction of classical, limit and logarithmic Frobenius manifolds: we need a Frobenius type structure and a section of the corresponding bundle such that the associated period map is invertible, in other words a *primitive* section, see for instance [24, Chapitre VII]. To get such objects, we look, following [11] and [18], for unfoldings of \mathbb{F}_w and $\overline{\mathbb{F}}_w$ (in the limit case), which can thus be seen as "initial data": they will be obtained from unfoldings of the Saito structures \mathcal{S}_w and $\overline{\mathcal{S}}_w$. In the best cases, but not always, we use the reconstruction method presented in *loc. cit.* to get *universal* unfoldings: the existence of a primitive section, hence of a Frobenius manifold, follows from this universality property. We show first and in this way that

- (1) the Frobenius type structure \mathbb{F}_w yields a Frobenius manifold on $\Delta \times (\mathbb{C}^{\mu-1}, 0)$, Δ denoting any open disc in M . We will use it to compare, using the arguments given in [11], the canonical Frobenius manifolds attached to the functions $F_x := F(\cdot, x)$, $x \in \Delta$, by the punctual construction given in [14];
- (2) the limit Frobenius type structure $\overline{\mathbb{F}}_w$ yields "limit" Frobenius manifolds, depending on the weights w_0, \dots, w_n . For instance, we get a universal unfolding only in the manifold case (*i.e* $w_0 = \dots = w_n = 1$): as a consequence of the universality, we obtain a unique, up to isomorphism, canonical *limit* Frobenius manifold. In the orbifold case, that is if there is a weight w_i greater or equal to two, we construct a limit Frobenius manifold for which the product is constant, but we loose any kind of unicity: our limit Frobenius type structure could produce other Frobenius manifolds, which can be difficult to compare.

This distinction between the manifold case and the orbifold case also appears in the construction of *logarithmic* Frobenius manifolds. For instance, in the manifold case, we show how our initial data \mathbb{F}_w yields more precisely, as before *via* one of its universal unfoldings, a logarithmic Frobenius manifold with logarithmic pole along $x = 0$ in the sense of [23]. This gives the logarithmic Frobenius manifold attached to \mathbb{P}^n in *loc. cit.* by a different method (Reichelt works directly with the whole Gromov-Witten potential; more generally, he constructs a logarithmic Frobenius manifold from the big quantum cohomology of any smooth manifold). In the orbifold case, our metric degenerates at the origin and we get only a logarithmic Frobenius manifold *without metric*. The construction of a logarithmic Frobenius manifold using this method is still an open problem. We also explain why Reichelt's construction does not work in the orbifold case.

The paper is organized as follows: we introduce the combinatorics and we define the Saito structures and the Frobenius type structures in section 2. The construction of the Saito structure attached to an orbifold (the A -model D -module) is done in section 3. It is explained in the case of the weighted projective spaces. Section 4 is devoted to the construction

of the B -model D -module and the main theorem is stated in section 5. We compute the limits of our structures in section 6 and we discuss the construction of Frobenius manifolds in section 7.

This paper is a revised version of the preprint [12] and supersedes it.

2. PRELIMINARIES

2.1. Combinatorics. Let w_0, w_1, \dots, w_n be positive integers and

$$F := \left\{ \frac{\ell}{w_i} \mid 0 \leq \ell \leq w_i - 1, 0 \leq i \leq n \right\}.$$

We denote by f_1, \dots, f_k the elements of F arranged in increasing order:

$$0 = f_1 < f_2 < \dots < f_k < f_{k+1} := 1.$$

For $f \in \mathbb{Q}$, we define

$$(1) \quad S_f := \{j \mid w_j f \in \mathbb{Z}\} \subset \{0, \dots, n\} \text{ and } m_i := \prod_{j \in S_{f_i}} w_j.$$

The *multiplicity*, denoted by d_i , of f_i is the positive integer defined by $d_i := \#S_{f_i}$. In particular we have $S_{f_1} = \{0, \dots, n\}$, $m_1 = w_0 \cdots w_n$ and $d_1 = n + 1$. Notice that

$$d_1 + \dots + d_k = w_0 + \dots + w_n := \mu.$$

Let $c_0, c_1, \dots, c_{\mu-1}$ be the sequence

$$\underbrace{f_1, \dots, f_1}_{d_1}, \underbrace{f_2, \dots, f_2}_{d_2}, \dots, \underbrace{f_k, \dots, f_k}_{d_k}$$

arranged in increasing order. It can be obtained as follows (see [14, p. 3]): define inductively the sequence $(a(k), i(k)) \in \mathbb{N}^{n+1} \times \{0, \dots, n\}$ by $a(0) = (0, \dots, 0)$, $i(0) = 0$ and

$$a(k+1) = a(k) + \mathbf{1}_{i(k)} \text{ where } i(k) := \min_j \{i \mid a(k)_i / w_i = \min_j a(k)_j / w_j\}.$$

In particular, $a(1) = (1, 0, \dots, 0)$, $a(n+1) = (1, \dots, 1)$, $a(\mu) = (1, w_1, \dots, w_n)$ and $\sum_{i=0}^n a(k)_i = k$. Then we have :

$$c_k = a(k)_{i(k)} / w_{i(k)}.$$

Lemma 2.1.1. *We have $c_0 = \dots = c_n = 0$, $c_{n+1} = \frac{1}{\max_i w_i}$ and $c_k + c_{\mu+n-k} = 1$ for $k \geq n+1$.*

Proof. See [14, p. 2]. □

Define now, for $k = 0, \dots, \mu - 1$, $\alpha_k := k - \mu c_k$.

Corollary 2.1.2. *We have $\alpha_0 = 0, \dots, \alpha_n = n$, $\alpha_{k+1} \leq \alpha_k + 1$ for all k ,*

$$\alpha_k + \alpha_{\mu+n-k} = n$$

for $k = n+1, \dots, \mu - 1$ and

$$\alpha_k + \alpha_{n-k} = n$$

for $k = 0, \dots, n$.

The α_k 's will give the *spectrum at infinity* of a certain regular function on the B-side (see section 4) and half of the *orbifold degree* on the A-side (see section 3.3.1). Notice that these numbers are integers if and only if $w_i \mid \mu$ for $i = 0, \dots, n$.

Example 2.1.3. Let $w_0 = 1$, $w_1 = 2$, $w_2 = 2$. We have :

- $\mu = 5$,

- $f_1 = 0$, $d_1 = 3$, $f_2 = \frac{1}{2}$, $d_2 = 2$, $S_{f_1} = \{0, 1, 2\}$ and $S_{f_2} = \{1, 2\}$,
- $a(0) = (0, 0, 0)$, $a(1) = (1, 0, 0)$, $a(2) = (1, 1, 0)$, $a(3) = (1, 1, 1)$, $a(4) = (1, 2, 1)$
- $c_0 = c_1 = c_2 = 0$, $c_3 = c_4 = \frac{1}{2}$ and $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = \frac{1}{2}$, $\alpha_4 = \frac{3}{2}$.

We will follow this example all along this paper.

2.2. Saito structures and Frobenius type structures.

Definition 2.2.1. Let M be a complex manifold, n be a positive integer. A *Saito structure of weight n^1* on $\mathbb{P}^1 \times M$ is a tuple (M, H, ∇, S, n) where

- H is a trivial bundle over $\mathbb{P}^1 \times M$,
- ∇ is a meromorphic, flat connection on H with poles along $\{0, \infty\} \times M$, logarithmic along $\{\infty\} \times M$, of order 1 along $\{0\} \times M$,
- S is a ∇ -flat, nondegenerate \mathbb{C} -bilinear form, satisfying

$$S : \mathcal{H} \times i^* \mathcal{H} \rightarrow z^n \mathcal{O}_{\mathbb{P}^1 \times M}$$

where \mathcal{H} is the sheaf of section of H , z is the coordinate on $\mathbb{P}^1 \setminus \{\infty\}$ and

$$i : \mathbb{P}^1 \times M \rightarrow \mathbb{P}^1 \times M$$

sends (z, \underline{t}) to $(-z, \underline{t})$.

Definition 2.2.2. Two Saito structures $(M_1, H_1, \nabla_1, S_1, n_1)$ and $(M_2, H_2, \nabla_2, S_2, n_2)$ are *isomorphic* if there exists an isomorphism $(\text{id}, \tau) : \mathbb{P}^1 \times M_1 \rightarrow \mathbb{P}^1 \times M_2$ and an isomorphism of vector bundles $\gamma : H_1 \rightarrow (\text{id}, \tau)^* H_2$ compatible with the connections and the metrics, *i.e* such that

- $\nabla_2^* \gamma(s) = \gamma(\nabla_1 s)$ for any section s of H_1 ,
- $S_2^*(\gamma(e), \gamma(f)) = S_1(e, f)$ for any sections e and f of H_1 (in particular $n_1 = n_2$),

∇_2^* (*resp.* S_2^*) denoting the connection (*resp.* the metric) on $(\text{id}, \tau)^* H_2$ induced by ∇_2 (*resp.* S_2).

Definition 2.2.3. A *Frobenius type structure*² on M is a tuple

$$(M, E, \nabla, R_0, R_\infty, \Phi, g)$$

where

- E is a locally free sheaf of \mathcal{O}_M -modules,
- R_0 and R_∞ are \mathcal{O}_M -linear endomorphisms of E ,
- $\Phi : E \rightarrow \Omega^1(M) \otimes E$ is a \mathcal{O}_M -linear map,
- g is a \mathcal{O}_M -bilinear form, symmetric and nondegenerate (a *metric*),
- ∇ is a connection on E

these objects satisfying the relations

$$\begin{aligned} \nabla^2 &= 0, \nabla(R_\infty) = 0, \Phi \wedge \Phi = 0, [R_0, \Phi] = 0, \\ \nabla(\Phi) &= 0, \nabla(R_0) + \Phi = [\Phi, R_\infty], \\ \nabla(g) &= 0, \Phi^* = \Phi, R_0^* = R_0, R_\infty + R_\infty^* = r \text{id} \end{aligned}$$

for a suitable constant r , $*$ denoting as above the adjoint with respect to g .

Remark 2.2.4. (1) A Saito structure on \mathbb{P}^1 (*i.e* $M = \{\text{point}\}$) will be denoted by (H, ∇, S, n) .
(2) A Frobenius type structure on a point is a tuple

$$(E, R_0, R_\infty, g)$$

where E is a finite dimensional vector space over \mathbb{C} , g is a symmetric and nondegenerate bilinear form on E , R_0 and R_∞ being two endomorphisms of E satisfying $R_0^* = R_0$ and

¹After K. Saito. It is sometimes called a $tr(TLEP)(n)$ -structure, see [19, Section 5.2]

²This terminology is borrowed from [18]

$R_\infty + R_\infty^* = r \text{ id}$ for a suitable complex number r , $*$ denoting the adjoint with respect to g .

◆

A Saito structure yields a Frobenius type structure (see for instance [24, VI, paragraphe 2]). Indeed, let (M, H, ∇, S, n) be a Saito structure on $\mathbb{P}^1 \times M$, $\sigma_1, \dots, \sigma_r$ be a basis of global sections of H . Define

- $E := H|_{\{0\} \times M}$ and $E_\infty := H|_{\{\infty\} \times M}$ (E and E_∞ are canonically isomorphic),
- $R_0[\sigma_i] := [\theta^2 \nabla_{\partial_\theta} \sigma_i]$, for $i = 1, \dots, r$,
- $g([\sigma_i], [\sigma_j]) := \theta^{-n} S(\sigma_i, \sigma_j)$ for $i, j = 1, \dots, r$,
- $\Phi_\xi[\sigma_i] := [\theta \nabla_\xi \sigma_i]$ for any vector field ξ on M , $[\]$ denoting the class in E .

The connection ∇ and the endomorphism R_∞ are defined analogously, using the restriction E_∞ : we put, with $\tau = z^{-1}$,

- $R_\infty[\sigma_i] := [\nabla_{\tau \partial_\tau} \sigma_i]$
- $\nabla_\xi[\sigma_i] := [\nabla_\xi \sigma_i]$.

Proposition 2.2.5 (see [24]). *The tuple $(M, E, R_0, R_\infty, \Phi, \nabla, g)$ is a Frobenius type structure on M .*

Notice that the characteristic relations of a Frobenius type structure is the counterpart of the integrability of the connection of the associated Saito structure.

3. A-MODEL

Let \mathcal{X} be a smooth Deligne-Mumford stack of finite type over \mathbb{C} of complex dimension n . In this section, we construct a Saito structure on $\mathbb{P}^1 \times M_A$ where $M_A := H_{\text{orb}}^*(\mathcal{X}, \mathbb{C})$ (a quantum D -module in the sense of [20]; a similar notion, called *semi-infinite variation of Hodge structure* is defined by Barannikov in [2] and [3]). This will be our big A -model D -module. We restrict it to $H^2(\mathcal{X}, \mathbb{C})$ and we quotient the result by an action of the Picard group of \mathcal{X} to get the small A -model D -module. Finally, we explain this construction for weighted projective spaces.

Our general references on orbifolds and orbifold cohomology will be [1], [6] and [7].

3.1. The big A -model D -module. First, we recall some basic facts about orbifold cohomology. The *inertia stack*, denoted by $\mathcal{IX} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$, is the fiber product over the two diagonal morphisms $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$. The inertia stack is a smooth Deligne-Mumford stack but different components will in general have different dimensions. The identity section gives an irreducible component which is canonically isomorphic to \mathcal{X} . This component is called *the untwisted sector*. All the other components are called *twisted sectors*. We thus have

$$\mathcal{IX} = \mathcal{X} \sqcup \bigsqcup_{v \in T} \mathcal{X}_v$$

where T parametrizes the set of components of the twisted sectors of \mathcal{IX} .

The orbifold cohomology of \mathcal{X} is defined, as vector space, by $H_{\text{orb}}^*(\mathcal{X}, \mathbb{C}) := H^*(\mathcal{IX}, \mathbb{C})$. We have

$$H_{\text{orb}}^*(\mathcal{X}, \mathbb{C}) = H^*(\mathcal{X}, \mathbb{C}) \oplus \bigoplus_{v \in T} H^*(\mathcal{X}_v, \mathbb{C}).$$

We will put $M_A := H_{\text{orb}}^*(\mathcal{X}, \mathbb{C})$ in what follows.

To define a grading on M_A , we associate to any $v \in T$ a rational number called the *age* of \mathcal{X}_v . A geometric point (x, g) in \mathcal{IX} is a point x of \mathcal{X} and $g \in \text{Aut}(x)$. Fix a point $(x, g) \in \mathcal{X}_v$. As g acts on the tangent space $T_x \mathcal{X}$, we have an eigenvalue decomposition of $T_x \mathcal{X}$. For

any $f \in [0, 1[$, we denote $(T_x \mathcal{X})_f$ the sub-vector space where g acts by multiplication by $\exp(2\sqrt{-1}\pi f)$. We define

$$\text{age}(v) := \sum_{f \in [0, 1[} f \cdot \dim_{\mathbb{C}}(T_x \mathcal{X})_f.$$

This rational number only depends on v . Let α_v be a homogeneous cohomology class of \mathcal{X}_v . We define the *orbifold degree* of α_v by

$$\deg^{\text{orb}}(\alpha_v) := \deg(\alpha_v) + 2 \text{age}(v).$$

Let ϕ_0, \dots, ϕ_N be a graded homogeneous basis of $H_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$ such that $\phi_0 \in H^0(\mathcal{X}, \mathbb{Q})$ and $\phi_1, \dots, \phi_s \in H^2(\mathcal{X}, \mathbb{Q})$. Notice that the cohomology classes ϕ_1, \dots, ϕ_s are in the cohomology of \mathcal{X} i.e in the cohomology of the untwisted sector. We denote also by ϕ_0, \dots, ϕ_N the image of these classes in $H_{\text{orb}}^*(\mathcal{X}, \mathbb{C})$. We will denote by $\underline{t} := (t_0, \dots, t_N)$ the coordinates of M_A associated to this basis.

3.1.1. The trivial bundle and the flat meromorphic connection. Let H^A be the trivial vector bundle over $\mathbb{P}^1 \times M_A$ whose fibers are $H_{\text{orb}}^*(\mathcal{X}, \mathbb{C})$. For $i \in \{0, \dots, N\}$, we see ϕ_i as a global section of the bundle H^A .

Define the vector field, called *the Euler vector field*,

$$\mathfrak{E} := \sum_{i=0}^N \left(1 - \frac{\deg^{\text{orb}}(\phi_i)}{2} \right) t_i \partial_i + \sum_{i=1}^s r_i \partial_i.$$

where the r_i are rational numbers determined by the equality $c_1(T\mathcal{X}) = \sum_{i=1}^s r_i \phi_i$ and ∂_i the vector field $\frac{\partial}{\partial t_i}$.

The big quantum product³ endows the vector bundle H^A with a product. We define a \mathcal{O}_{M_A} -linear homomorphism which will turn out to be an Higgs field (i.e. $\Phi \wedge \Phi = 0$ see Proposition 3.1.1)

$$\Phi : TM_A \rightarrow \text{End}(H^A) \text{ by } \Phi(\partial_i) = \phi_i \bullet_{\underline{t}}.$$

In coordinates, we have

$$\Phi = \sum_{i=0}^N \Phi^{(i)}(\underline{t}) dt_i$$

where $\Phi^{(i)}(\underline{t})$ is the endomorphism $\phi_i \bullet_{\underline{t}}$.

Define, on the trivial bundle H^A , the connection

$$\nabla^A := d_{M_A} + d_{\mathbb{P}^1} - \frac{1}{z} \pi^* \Phi + \left(\frac{1}{z} \Phi(\mathfrak{E}) + R_{\infty} \right) \frac{dz}{z}$$

where $\pi : \mathbb{P}^1 \times M_A \rightarrow M_A$ is the projection and R_{∞} is the semi-simple endomorphism whose matrix in the basis (ϕ_i) is

$$R_{\infty} = \text{Diag} \left(\frac{\deg^{\text{orb}}(\phi_0)}{2}, \dots, \frac{\deg^{\text{orb}}(\phi_N)}{2} \right).$$

The proposition below is well-known to the specialists, and already stated in [20], but we did not find a complete proof of it in the literature. Some parts and ideas can be found in [24], [19], [21] and [9].

Proposition 3.1.1. *The meromorphic connection ∇^A is flat.*

³Usually, working on quantum cohomology, one has either to add the Novikov ring (see section 8.1.3 of [9]) or to assume that the quantum product converges on some open of M_A (see Assumption 2.1 in [20]). But we will mainly consider the small quantum product of weighted projective spaces, for which the convergence problems are solved.

Proof. Set $R_0(\underline{t}) := \Phi(\mathfrak{E})$ which is the endomorphism of E defined by

$$\Phi(\mathfrak{E}) = \sum_{i=0}^N \left(1 - \frac{\deg^{\text{orb}}(\phi_i)}{2}\right) t_i \phi_i \bullet_{\underline{t}} + \sum_{i=1}^s r_i \phi_i \bullet_{\underline{t}}.$$

The flatness of ∇^A is then equivalent to the following equalities :

$$(2) \quad \frac{\partial \Phi^{(i)}(\underline{t})}{\partial t_j} - \frac{\partial \Phi^{(j)}(\underline{t})}{\partial t_i} = 0 \text{ for } i, j \in \{0, \dots, N\}$$

$$(3) \quad [\Phi^{(i)}(\underline{t}), \Phi^{(j)}(\underline{t})] = 0 \text{ for } i, j \in \{0, \dots, N\}$$

$$(4) \quad [R_0(\underline{t}), \Phi^{(i)}(\underline{t})] = 0 \text{ for } i \in \{0, \dots, N\}$$

$$(5) \quad \frac{\partial R_0(\underline{t})}{\partial t_i} - \Phi^{(i)}(\underline{t}) = [\Phi^{(i)}(\underline{t}), R_\infty] \text{ for } i \in \{0, \dots, N\}$$

Let $F_0(\underline{t})$ be the Gromov-Witten potentiel of genus 0. We have

$$(6) \quad \Phi(\partial_i)(\phi_j) = \phi_i \bullet_{\underline{t}} \phi_j = \sum_{k=0}^N \Phi_{jk}^{(i)} \phi_k.$$

and thus, by [9, p.231],

$$(7) \quad \Phi(\partial_i)(\phi_j) = \sum_{\ell=0}^N \partial_i \partial_j \partial_\ell F_0(\underline{t}) \phi^\ell = \sum_{\ell=0}^N \sum_{k=0}^N \partial_i \partial_j \partial_\ell F_0 g^{\ell k} \phi_k$$

where ϕ^ℓ is the orbifold Poincaré dual of ϕ_ℓ and the matrix $(g^{\ell k})$ is the inverse matrix of the matrix of the orbifold Poincaré duality on M_A . From (6) and (7), we deduce that, for any $i, j \in \{0, \dots, N\}$, we have

$$\Phi_{jk}^{(i)}(\underline{t}) = \sum_{\ell} \partial_i \partial_j \partial_\ell F_0(\underline{t}) g^{\ell k}.$$

This implies the equalities (2). The second equalities (3) follows from the associativity of the quantum product (see §8.4 of [9]). We have

$$\begin{aligned} R_0(\underline{t}) \Phi^{(i)}(\underline{t})(\phi_j) &= \sum_{k=0}^N \left(1 - \frac{\deg^{\text{orb}}(\phi_k)}{2}\right) t_k \phi_k \bullet_{\underline{t}} (\phi_i \bullet_{\underline{t}} \phi_j) + \sum_{k=1}^s r_k \phi_k \bullet_{\underline{t}} (\phi_i \bullet_{\underline{t}} \phi_j) \\ \Phi^{(i)}(\underline{t}) R_0(\underline{t})(\phi_j) &= \sum_{k=0}^N \left(1 - \frac{\deg^{\text{orb}}(\phi_k)}{2}\right) t_k \phi_i \bullet_{\underline{t}} (\phi_k \bullet_{\underline{t}} \phi_j) + \sum_{k=1}^s r_k \phi_i \bullet_{\underline{t}} (\phi_k \bullet_{\underline{t}} \phi_j) \end{aligned}$$

hence the equalities (4) follows from the associativity and commutativity of the quantum product. Let us prove now the fourth equalities (5). We have

$$(8) \quad [\Phi^{(j)}, R_\infty](\phi_j) = \frac{\deg^{\text{orb}}(\phi_j)}{2} \phi_i \bullet_{\underline{t}} \phi_j - \sum_{k=0}^N \partial_i \partial_j \partial_k F_0(\underline{t}) \left(n - \frac{\deg^{\text{orb}}(\phi_k)}{2}\right) \phi^k.$$

because $\deg^{\text{orb}}(\phi^k) + \deg^{\text{orb}}(\phi_k) = 2n$. On the other hand, using (2), we deduce that

$$(9) \quad \frac{\partial R_0(\underline{t})}{\partial t_i} - \Phi^{(i)}(\underline{t}) = \mathfrak{E} \cdot \Phi^{(i)}(\underline{t}) - \frac{\deg^{\text{orb}}(\phi_i)}{2} \Phi^{(i)}(\underline{t}).$$

Moreover the Euler vector field \mathfrak{E} satisfies the following properties (see [21, p.24])

$$(10) \quad \mathfrak{E} \partial_i \partial_j \partial_k F_0(\underline{t}) = \partial_i \partial_j \partial_k \mathfrak{E} F_0(\underline{t}) + \partial_i \partial_j [\mathfrak{E}, \partial_k] F_0(\underline{t}) + \partial_i [\mathfrak{E}, \partial_j] \partial_k F_0(\underline{t}) + [\mathfrak{E}, \partial_i] \partial_j \partial_k F_0(\underline{t})$$

$$(11) \quad [\mathfrak{E}, \partial_i] = \left(\frac{\deg^{\text{orb}}(\phi_i)}{2} - 1 \right) \partial_i$$

$$(12) \quad \mathfrak{E}F_0(\underline{t}) = (3 - \dim_{\mathbb{C}} \mathcal{X})F_0(\underline{t}).$$

Using the relations (10), (11) and (12), we get

$$(13) \quad \begin{aligned} & \mathfrak{E} \cdot \Phi^{(i)}(\phi_j) = \mathfrak{E}(\phi_i \bullet_{\underline{t}} \phi_j) \\ &= \sum_{k=0}^N \left(\frac{\deg^{\text{orb}}(\phi_i)}{2} + \frac{\deg^{\text{orb}}(\phi_j)}{2} + \frac{\deg^{\text{orb}}(\phi_k)}{2} - n \right) \partial_i \partial_j \partial_k F_0(\underline{t}) \phi^k \\ &= \left(\frac{\deg^{\text{orb}}(\phi_i)}{2} + \frac{\deg^{\text{orb}}(\phi_j)}{2} \right) \phi_i \bullet_{\underline{t}} \phi_j + \sum_{k=0}^N \left(\frac{\deg^{\text{orb}}(\phi_k)}{2} - n \right) \partial_i \partial_j \partial_k F_0(\underline{t}) \phi^k \end{aligned}$$

Putting together (8), (9) and (13), we deduce the last equalities (5). \square

Remark 3.1.2. The connection $d_{M_A} - \frac{1}{z}\pi^*\Phi$ is flat (see [9, §8.4 and §10.4]): this is equivalent to equalities (2) and (3). \blacklozenge

3.1.2. *The pairing.* The vector space $H_{\text{orb}}^*(\mathcal{X}, \mathbb{C})$ is endowed with a nondegenerate pairing which is called the orbifold Poincaré pairing (see [7]). We denote it by $\langle \cdot, \cdot \rangle$. It satisfies the following homogeneity property:

$$(14) \quad \text{if } \langle \phi_i, \phi_j \rangle \neq 0 \text{ then } \deg^{\text{orb}}(\phi_i) + \deg^{\text{orb}}(\phi_j) = 2n.$$

We define a pairing S^A on the global sections ϕ_0, \dots, ϕ_N of H^A by

$$S^A(\phi_i, \phi_j) := z^n \langle \phi_i, \phi_j \rangle.$$

and we extend it by linearity using the rules

$$(15) \quad a(z, \underline{t})S^A(\cdot, \cdot) = S^A(a(z, \underline{t})\cdot, \cdot) = S^A(\cdot, a(-z, \underline{t})\cdot)$$

for any $a(z, \underline{t}) \in \mathcal{O}_{\mathbb{P}^1 \times M_A}$.

Proposition 3.1.3. *The pairing $S^A(\cdot, \cdot)$ is nondegenerate, $(-1)^n$ symmetric and ∇^A -flat.*

Proof. As the orbifold Poincaré duality is nondegenerate, the pairing S^A is nondegenerate and $(-1)^n$ -symmetric by (15). The ∇^A -flatness is equivalent to

$$(16) \quad z\partial_z S^A(\phi_i, \phi_j) = S^A(\nabla_{z\partial_z}^A \phi_i, \phi_j) + S^A(\phi_i, \nabla_{z\partial_z}^A \phi_j)$$

$$(17) \quad \partial_k S^A(\phi_i, \phi_j) = S^A(\nabla_{\partial_k}^A \phi_i, \phi_j) + S^A(\phi_i, \nabla_{\partial_k}^A \phi_j)$$

Using the rules (15), we have

$$\begin{aligned} z\partial_z S^A(\phi_i, \phi_j) &= nS^A(\phi_i, \phi_j) \\ S^A(z\nabla_{\partial_z}^A \phi_i, \phi_j) &= \frac{1}{z}S^A(\Phi(\mathfrak{E})(\phi_i), \phi_j) + S^A(R_{\infty}\phi_i, \phi_j) \\ S^A(\phi_i, \nabla_{z\partial_z}^A \phi_j) &= -\frac{1}{z}S^A(\phi_i, \Phi(\mathfrak{E})(\phi_j)) + S^A(\phi_i, R_{\infty}\phi_j) \end{aligned}$$

We denote by R_{∞}^* the adjoint of R_{∞} with respect to $S^A(\cdot, \cdot)$. The following equalities

$$(18) \quad \langle \phi_k \bullet_{\underline{t}} \phi_i, \phi_j \rangle = \langle \phi_i, \phi_k \bullet_{\underline{t}} \phi_j \rangle$$

$$(19) \quad R_{\infty} + R_{\infty}^* = n \text{ id}$$

imply (16). The left hand side of (17) vanishes because $S^A(\phi_i, \phi_j)$ does not depends on the coordinates \underline{t} . The equalities (18) implies that the right hand side also vanishes. \square

From propositions 3.1.1 and 3.1.3 we get

Corollary 3.1.4. *The tuple $(M_A, H^A, \nabla^A, S^A, n)$ is a Saito structure on $\mathbb{P}^1 \times M_A$.*

Definition 3.1.5. The Saito structure $(M_A, H^A, \nabla^A, S^A, n)$ is called *the big A -model D -module associated to \mathcal{X}* .

Remark 3.1.6. Iritani defines also a A -model D -module [20, definition 2.2] and his definition is very closed from ours (compare with remark 4.5.3). There are some mild differences: the first one is that Iritani considers the opposite of our Higgs field and, in order to identify H^A with π^*TM_A , he uses $\phi_i \mapsto \partial_i$ whereas we use $\phi_i \mapsto -\partial_i$ (we choose the minus sign because usually the infinitesimal period map on the B -side is defined with a minus sign). The second one is that Iritani considers the matrix $R_\infty - \frac{n}{2} \text{id}$ which has symmetric eigenvalues with respect to 0 (in our case, the eigenvalues are symmetric with respect to $n/2$). ♦

3.2. The small A -model D -module. On a manifold X , the small quantum product is the restriction of the big one to $H^2(X, \mathbb{C})$, that is $\bullet_{\underline{t}}$ where $\underline{t} \in H^2(X, \mathbb{C})$. The classes in $H^2(X, \mathbb{C})$ play a special role because they satisfy the divisor axiom for Gromov-Witten invariants. For orbifolds, the divisor axiom works only for classes in the second cohomology group of the untwisted sector (see Theorem 8.3.1 of [1]), that is $H^2(\mathcal{X}, \mathbb{C})$ (and not $H_{\text{orb}}^2(\mathcal{X}, \mathbb{C})$).

3.2.1. Restriction of the big A -model D -module. We first restrict the big A -model D -module $(M_A, H^A, \nabla^A, S^A, n)$ to $M_A^{\text{sm}} := H^2(\mathcal{X}, \mathbb{C})$ and we get a Saito structure on $\mathbb{P}^1 \times M_A^{\text{sm}}$ denoted by

$$(M_A^{\text{sm}}, H^{A, \text{sm}}, \nabla^{A, \text{sm}}, S^{A, \text{sm}}, n).$$

Let $\underline{t}^{\text{sm}} := (t_1, \dots, t_s)$ be the coordinates on M_A^{sm} . The restricted connection is

$$(20) \quad \nabla^{A, \text{sm}} = d_{M_A^{\text{sm}}} + d_{\mathbb{P}^1} - \frac{1}{z} \Phi^{\text{sm}} + \left(\frac{1}{z} \Phi^{\text{sm}}(\mathfrak{E}^{\text{sm}}) + R_\infty \right) \frac{dz}{z}$$

where Φ^{sm} (resp. \mathfrak{E}^{sm}) is the restriction of Φ (resp. \mathfrak{E}) on TM_A^{sm} . In coordinates, we have

$$\Phi^{\text{sm}} = \sum_{i=1}^s \Phi^{(i)}(\underline{t}^{\text{sm}}) dt_i \text{ and } \mathfrak{E}^{\text{sm}} = \sum_{i=1}^s r_i \partial_i.$$

Notice that \mathfrak{E}^{sm} is uniquely determined by $c_1(T\mathcal{X})$ and that $\Phi^{\text{sm}}(\mathfrak{E}^{\text{sm}})$ is the small quantum multiplication by $c_1(T\mathcal{X})$.

3.2.2. An action of $\text{Pic}(\mathcal{X})$. For manifolds, the quantum product is equivariant with respect to the action of the Picard group. In this section, following Iritani [20], we extend this action to the orbifold case.

Let L be a line bundle on the orbifold \mathcal{X} . For any point $x \in \mathcal{X}$, we have an action of $\text{Aut}(x)$ on the fiber of L at x denoted by L_x that is an element on $GL(L_x)$. Hence, for any point $(x, g) \in \mathcal{X}_v \subset \mathcal{IX}$, we have an element $f_v(L) \in \mathbb{Q} \cap [0, 1[$ such that the action of g on L_x is the multiplication by $e^{2\pi\sqrt{-1}\pi f_v(L)}$. The rational number $f_v(L)$ depends only of $v \in T$ (see [1, section 7]).

Remark 3.2.1. If \mathcal{X} is a toric orbifold, then we have $\mathcal{X} = [Z/G]$ where $G := \text{Hom}(\text{Pic}(\mathcal{X}), \mathbb{C}^*)$ and Z is a quasi-affine variety in some \mathbb{C}^m (cf. [4] and [15] for a more precise definition). The inertia stack is parametrized by a finite subset T of G . A line bundle L on \mathcal{X} is given by a character χ_L of G (see [15]). In this special case, $f_v(L)$ is defined by the equality $\chi_L(v) = e^{2\pi\sqrt{-1}\pi f_v(L)}$. ♦

We define now an action of $\text{Pic}(\mathcal{X})$ on $(M_A^{\text{sm}}, H^{A, \text{sm}}, \nabla^{A, \text{sm}}, S^{A, \text{sm}}, n)$ as follows:

(1) on the fibers of $H^{A,\text{sm}}$, for $\alpha \oplus \bigoplus_{v \in T} \alpha_v \in H^*(\mathcal{X}, \mathbb{C}) \oplus \bigoplus_{v \in T} H^*(\mathcal{X}_v, \mathbb{C})$ the action is given by

$$(21) \quad L \cdot \left(\alpha \oplus \bigoplus_{v \in T} \alpha_v \right) = \alpha \oplus \bigoplus_{v \in T} e^{2\pi\sqrt{-1}f_v(L)} \alpha_v$$

(2) on $M_A^{\text{sm}} = H^2(\mathcal{X}, \mathbb{C})$ we define

$$(22) \quad \text{Pic}(\mathcal{X}) \times H^2(\mathcal{X}, \mathbb{C}) \longrightarrow H^2(\mathcal{X}, \mathbb{C})$$

$$\left(L, \sum_{i=1}^s t_i \phi_i \right) \longmapsto \left(\sum_{i=1}^s t_i \phi_i \right) - 2\pi\sqrt{-1}c_1(L) = \sum_{i=1}^s (t_i - 2\pi\sqrt{-1}L_i) \phi_i$$

where $c_1(L) = \sum_{i=1}^s L_i \phi_i$.

Proposition 3.2.2 (see proposition 2.3 of [20]). (1) *The small quantum product is equivariant with respect to this action: for any classes $\alpha, \beta \in H_{\text{orb}}^*(\mathcal{X}, \mathbb{C})$, for any point $\underline{t}^{\text{sm}} \in H^2(\mathcal{X}, \mathbb{C})$ and for any $L \in \text{Pic}(\mathcal{X})$, we have*

$$(L \cdot \alpha) \bullet_{L, \underline{t}^{\text{sm}}} (L \cdot \beta) = L \cdot (\alpha \bullet_{\underline{t}^{\text{sm}}} \beta).$$

(2) *The pairing $S^{A,\text{sm}}(\cdot, \cdot)$ is invariant with respect to this action.*

Proof. Recall that we denote by ϕ^i the Poincaré dual of ϕ_i . By definition of the small quantum product, we have

$$(L \cdot \alpha) \bullet_{L, \underline{t}^{\text{sm}}} (L \cdot \beta) = \sum_{d \in H_2(\mathcal{X}, \mathbb{Q})} \sum_{i=0}^N \langle L \cdot \alpha, L \cdot \beta, \phi_i \rangle_{0,3,d} \phi^i e^{\int_d (\underline{t}^{\text{sm}} - 2\pi\sqrt{-1}c_1(L))}.$$

By definition of the Poincaré duality, we have that $L \cdot \phi^i = L^{-1} \cdot \phi_i$. Using the proof of Proposition 2.3 in [20], we deduce that

$$\begin{aligned} (L \cdot \alpha) \bullet_{L, \underline{t}^{\text{sm}}} (L \cdot \beta) &= \sum_{d \in H_2(\mathcal{X}, \mathbb{Q})} \sum_{i=0}^N \langle L \cdot \alpha, L \cdot \beta, L \cdot \phi_i \rangle_{0,3,d} (L \cdot \phi^i) e^{\int_d (\underline{t}^{\text{sm}} - 2\pi\sqrt{-1}c_1(L))} \\ &= \sum_{d \in H_2(\mathcal{X}, \mathbb{Q})} \sum_{i=0}^N \langle \alpha, \beta, \phi_i \rangle_{0,3,d} (L \cdot \phi^i) e^{\int_d \underline{t}^{\text{sm}}} \\ &= L \cdot (\alpha \bullet_{\underline{t}^{\text{sm}}} \beta). \end{aligned}$$

For the second statement, we show that for any $\alpha_v \in H^*(\mathcal{X}_v, \mathbb{C})$, for any $\alpha_w \in H^*(\mathcal{X}_w, \mathbb{C})$ and for any $L \in \text{Pic}(\mathcal{X})$, we have :

$$S(L \cdot \alpha_v, L \cdot \alpha_w) = S(\alpha_v, \alpha_w).$$

We have that $S(\alpha_v, \alpha_w) \neq 0$ implies that the involution of $I\mathcal{X}$ sending $(x, g) \rightarrow (x, g^{-1})$ maps \mathcal{X}_v to \mathcal{X}_w (see the definition of the orbifold Poincaré duality in [7]). This implies that $f_v(L) + f_w(L) \in \{0, 1\}$. Hence, we have

$$S(L \cdot \alpha_v, L \cdot \alpha_w) = e^{2\pi\sqrt{-1}(f_v(L)+f_w(L))} S(\alpha_v, \alpha_w) = S(\alpha_v, \alpha_w).$$

□

Remark 3.2.3. By the divisor axiom, the variables corresponding to $H^2(\mathcal{X}, \mathbb{C})$ appear as exponential in the genus 0 Gromov-Witten potential. For $i \in \{1, \dots, s\}$, we have indeed terms of the form $e^{t_i \int_{\beta} \phi_i}$ for $\beta \in H_2(\mathcal{X}, \mathbb{Q})$ and the action above acts on these terms as follows

$$(23) \quad L \cdot e^{\sum_{i=1}^s t_i \int_{\beta} \phi_i} = e^{\sum_{i=1}^s t_i \int_{\beta} \phi_i} e^{-2\pi\sqrt{-1} \int_{\beta} c_1(L)}.$$

Since, for orbifolds, the classes β and the Chern classes are rational, the action of the Picard group is not trivial. So the multiplication by $\exp\left(-2\pi\sqrt{-1}\int_{\beta}c_1(L)\right)$ has to be corrected by a natural action on the fibers of $H^{A,\text{sm}}$ on the twisted cohomology classes in order to get the proposition above. For manifolds, the homology class β and the Chern classes are integral, hence the action (23) is trivial: the quantum product for manifold is invariant with respect to this action. \blacklozenge

3.2.3. The quotient structure. It follows from proposition 3.2.2 that the Saito structure $(M_A^{\text{sm}}, H^{A,\text{sm}}, S^{A,\text{sm}}, n)$ is $\text{Pic}(\mathcal{X})$ -equivariant. Hence, it defines a quotient Saito structure denoted by

$$\mathcal{S}^A := (\mathcal{M}_A, \tilde{H}^{A,\text{sm}}, \tilde{\nabla}^{A,\text{sm}}, \tilde{S}^{A,\text{sm}}, n)$$

where

$$\mathcal{M}_A := H^2(\mathcal{X}, \mathbb{C}) / \text{Pic}(\mathcal{X}) \simeq (\mathbb{C}^*)^s.$$

Corollary 3.2.4. *The tuple \mathcal{S}^A is a Saito structure on $\mathbb{P}^1 \times \mathcal{M}_A$.*

Definition 3.2.5. The Saito structure $(\mathcal{M}_A, \tilde{H}^{A,\text{sm}}, \tilde{\nabla}^{A,\text{sm}}, \tilde{S}^{A,\text{sm}}, n)$ is called the *small A-model D-module*.

Remark 3.2.6. For $i \in \{0, \dots, N\}$, ϕ_i is a global section of $H^{A,\text{sm}}$. We have

$$\phi_i \text{ is a global section of } \tilde{H}^{A,\text{sm}} \iff L \cdot \phi_i = \phi_i, \forall L \in \text{Pic}(\mathcal{X}).$$

We deduce that the classes ϕ_i in the cohomology of the untwisted sector are global sections of $\tilde{H}^{A,\text{sm}}$. Notice that if s_1 and s_2 are global sections of $\tilde{H}^{A,\text{sm}}$, then so is $s_1 \bullet_{\tilde{t}^{\text{sm}}} s_2$. To find a basis of global section of $\tilde{H}^{A,\text{sm}}$, we will look for sections of the kind $s_1 \bullet_{\tilde{t}^{\text{sm}}} s_2$. \blacklozenge

For $i \in \{1, \dots, s\}$, we put $q_i := \exp(t_i)$. However, the $\underline{q} := (q_1, \dots, q_s)$ are not coordinates on \mathcal{M}_A because, for $L \in \text{Pic}(\mathcal{X})$, we have

$$(24) \quad L \cdot q_i = q_i e^{-2\pi\sqrt{-1}L_i}$$

where L_i are rational numbers defined by $c_1(L) = \sum_{i=1}^s L_i \phi_i \in H^2(\mathcal{X}, \mathbb{Q})$, see (22). However if we choose $\mathcal{L}_1, \dots, \mathcal{L}_s$ as generators of the free part⁴ of $\text{Pic}(\mathcal{X})$ and put $\phi_i := c_1(\mathcal{L}_i)$, then the L_i 's are now integers *i.e.*, (q_1, \dots, q_s) are coordinates⁵ on \mathcal{M}_A . In such a choice of coordinates on \mathcal{M}_A , the connection $\tilde{\nabla}^{A,\text{sm}}$ is given by

$$(25) \quad \tilde{\nabla}^{A,\text{sm}} = d_{\mathcal{M}_A} + d_{\mathbb{P}^1} - \frac{1}{z} \tilde{\Phi}^{\text{sm}} + \left(\frac{1}{z} \tilde{\Phi}^{\text{sm}}(\tilde{\mathfrak{E}}^{\text{sm}}) + R_{\infty} \right) \frac{dz}{z}$$

where

$$\tilde{\Phi}^{\text{sm}} = \sum_{i=1}^s \Phi^{(i)} \frac{dq_i}{q_i} \text{ and } \tilde{\mathfrak{E}}^{\text{sm}} = \sum_{i=1}^s r_i q_i \frac{\partial}{\partial q_i}.$$

Remark 3.2.7. We first restrict the big A-model D-module $(M_A, H^A, \nabla^A, S^A, n)$ to $\mathbb{P}^1 \times H^2(\mathcal{X}, \mathbb{C})$ and then we quotient it by the action of $\text{Pic}(\mathcal{X})$. In [20], Iritani defines a global action, called Galois action, of $\text{Pic}(\mathcal{X})$ on $(M_A, H^A, \nabla^A, S^A, n)$, giving a Saito structure on $M_A / \text{Pic}(\mathcal{X})$. If we restrict it to $\mathcal{M}_A = H^2(\mathcal{X}, \mathbb{C}) / \text{Pic}(\mathcal{X})$ we get the small A-model D-module above. \blacklozenge

⁴Observe that the first Chern class of a torsion line bundle vanishes.

⁵For manifolds, the situation is easier because one can choose ϕ_i as an integer cohomology class. Since $c_1(L)$ is an integer cohomology class, the L_i 's are integers.

3.3. The small A -model D -module for weighted projective spaces. We describe in this section the small A -model D -module

$$\mathcal{S}_w^A = (\mathcal{M}_A, \tilde{H}^{A,\text{sm}}, \tilde{\nabla}^{A,\text{sm}}, \tilde{S}^{A,\text{sm}}, n)$$

associated with the weighted projective space $\mathbb{P}(w) := \mathbb{P}(w_0, \dots, w_n)$, where w_0, \dots, w_n are positive integers with $w_0 = 1$. The index w recalls these weights.

3.3.1. The toric description. We use here the notations and the definitions given in section 2.1. Recall that we assume $w_0 = 1$. We follow the definition of [8] for weighted projective spaces, that is with negative weights,

$$(26) \quad \mathbb{P}(w_0, w_1, \dots, w_n) := [\mathbb{C}^{n+1} - \{0\}]/\mathbb{C}^*$$

where the action is given by $\lambda(x_0, \dots, x_n) := (\lambda^{-w_0}x_0, \dots, \lambda^{-w_n}x_n)$. It is a toric Deligne-Mumford stacks in the sense of [15] and [4]. Its stacky fan is given by

- the lattice $N := \mathbb{Z}^n$.
- the morphism $\beta : \mathbb{Z}^{n+1} \rightarrow N$ that sends the canonical basis e_i to $(0, \dots, 0, 1, 0, \dots, 0)$ and e_0 to $(-w_1, \dots, -w_n)$.
- the fan Σ in N is the complete fan where the rays are generated by $\beta(e_i)$.

Remark 3.3.1. (1) The Picard group of $\mathbb{P}(w)$ is \mathbb{Z} and it is generated by the line bundle $\mathcal{O}(1)$.

(2) For $i \in \{0, \dots, n\}$, each $\beta(e_i)$ corresponds to a toric divisor D_i . This toric divisor is simply the canonical inclusion of $\mathbb{P}(w_0, \dots, \widehat{w_i}, \dots, w_n) \hookrightarrow \mathbb{P}(w)$. The line bundle associated to the toric divisor D_i is $\mathcal{O}(w_i)$. The situation when $w_0 = 1$ is particularly nice, because the toric divisor D_0 is $\mathcal{O}(1)$ which generates the Picard group. We denote by $P := c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(w), \mathbb{Q}) \subset H_{\text{orb}}^2(\mathbb{P}(w), \mathbb{C})$. ♦

For any subset $I = \{i_1, \dots, i_\ell\} \subset \{0, \dots, n\}$, we put $\mathbb{P}(w_I) := \mathbb{P}(w_{i_1}, \dots, w_{i_\ell})$. Recall the sets F and S_f defined in (1). Following [22] and [8], the inertia stack is

$$\mathcal{IP}(w) := \bigsqcup_{f \in F} \mathbb{P}(w_{S_f})$$

For any $f \in F$, denote by $\mathbf{1}_f$ the image of the cohomology class $\mathbf{1} \in H^0(\mathbb{P}(w_{S_f}), \mathbb{C})$ in $H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$. A basis of the orbifold cohomology $H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$, which is a \mathbb{C} -vector space of dimension μ , is given by the elements

$$(27) \quad \mathbf{1}_{f_i} P^j := \mathbf{1}_{f_i} \cup_{\text{orb}} \overbrace{P \cup_{\text{orb}} \dots \cup_{\text{orb}} P}^{j\text{-times}}, \text{ for } i \in \{1, \dots, k\} \text{ and } j \in \{0, \dots, d_i - 1\}.$$

The orbifold degree is now defined by

$$\deg^{\text{orb}} \mathbf{1}_{f_i} P^j := 2j + 2 \sum_{k=0}^n \{-w_k f_i\}$$

where $\{r\} := r - \lfloor r \rfloor$ is the fractional part of r . The orbifold Poincaré duality (see [22]) is given by

$$(28) \quad \langle \mathbf{1}_{f_i} P^k, \mathbf{1}_{f_j} P^\ell \rangle = \begin{cases} 1/m_i & \text{if } f_i + f_j \in \mathbb{N} \text{ and } k + \ell = d_i - 1 \\ 0 & \text{otherwise} \end{cases}$$

where $m_i = \prod_{j \in S_{f_i}} w_j$ (see (1)). Notice that if $f_i + f_j \in \mathbb{N}$ then $S_{f_i} = S_{f_j}$ so that the right hand side of (28) is symmetric in i and j .

3.3.2. *Description of the small A-model D-module.* Let t_1 be the coordinate on $H^2(\mathbb{P}(w), \mathbb{C})$, $q := \exp(t_1)$ and $C(q)$ be the matrix of the endomorphism P_{\bullet_q} of $H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$ in the basis $(\mathbf{1}_{f_i} P^j)$. This matrix is computed in [8] (see also [17]): we have

$$C(q) := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_\mu q^{1-c_{\mu-1}} \\ a_1 q^{c_1-c_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 q^{c_2-c_1} & 0 & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a_{\mu-1} q^{c_{\mu-1}-c_{\mu-2}} & 0 \end{pmatrix}$$

where

$$(29) \quad a_i := \begin{cases} 1/m_j & \text{if } i = d_1 + \cdots + d_j \\ 1 & \text{otherwise.} \end{cases}$$

Following the remark 3.2.6, we define, for $i \in \{0, \dots, \mu-1\}$,

$$(P^{\bullet_q})^i := \underbrace{P_{\bullet_q} \cdots P_{\bullet_q}}_{i \text{ times}} \text{ with } (P^{\bullet_q})^0 := \mathbf{1}_{f_1}.$$

Lemma 3.3.2 (See [8]). (1) *We have*

$$(30) \quad (P^{\bullet_q})^i = q^{c_i} s_i \mathbf{1}_{c_i} P^{r(i)}$$

where $r(i) := \#\{k \mid k < i \text{ and } c_k = c_i\}$ and $s_i = \prod_{k=0}^n w_k^{-[c_i w_k]}$. In particular, for each $q \neq 0$, the cohomology classes $((P^{\bullet_q})^i)_{0 \leq i \leq \mu-1}$ form a basis of the vector space $H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$.

(2) *For every i , $\deg^{\text{orb}}(P^{\bullet_q})^i = \deg^{\text{orb}} \mathbf{1}_{c_i} P^{r(i)}$.*

The following proposition refines the remark 3.2.6 for weighted projective spaces.

Proposition 3.3.3. *The Picard group $\text{Pic}(\mathbb{P}(w))$ acts on the two basis $(\mathbf{1}_{f_i} P^j)$ and $((P^{\bullet_q})^i)$ of $H_{\text{orb}}^*(\mathbb{P}(w))$ via the following formulas:*

$$\mathcal{O}(d) \cdot \mathbf{1}_f P^k = e^{-2\pi\sqrt{-1}df} \mathbf{1}_f P^k \text{ and } \mathcal{O}(d) \cdot (P^{\bullet_q})^i = (P^{\bullet_{\mathcal{O}(d) \cdot q}})^i.$$

for any $d \in \mathbb{Z}$. For $r \in \mathbb{Q}$, we have also $\mathcal{O}(d) \cdot q^r = q^r e^{-2\pi\sqrt{-1}dr}$.

Proof. Because we take the definition of weighted projective spaces with negative weights (see Formula (26)), the line bundle $\mathcal{O}(d)$ corresponds to the character $\chi : \mathbb{C}^* \rightarrow \mathbb{C}^*$ which sends $z \rightarrow z^{-d}$. Using remark 3.2.1, the action of $\mathcal{O}(d)$ on $\mathbf{1}_f P^k$ follows from the definition of the action (see formula (21)). For the action on q , it follows from the definition (see formula (22) and (24)). The action on $(P^{\bullet_q})^i$ follows from proposition 3.2.2. \square

Remark 3.3.4. From (30), we put $s(q) := (P^{\bullet_q})^i = q^{c_i} s_i \mathbf{1}_{c_i} P^{r(i)}$. We have

$$\begin{aligned} s(\mathcal{O}(d) \cdot q) &= (\mathcal{O}(d) \cdot q^{c_i}) s_i \mathbf{1}_{c_i} P^{r(i)} \\ &= q^{c_i} e^{-2\pi\sqrt{-1}dc_i} s_i \mathbf{1}_{c_i} P^{r(i)} \\ &= q^{c_i} s_i (\mathcal{O}(d) \cdot \mathbf{1}_{c_i} P^{r(i)}) \\ &= \mathcal{O}(d) \cdot s(q). \end{aligned}$$

As expected from remark 3.2.6, for $i \in \{0, \dots, N\}$, the section $(P^{\bullet_q})^i$ is a $\text{Pic}(\mathbb{P}(w))$ -equivariant section, hence it induces a global section of the bundle $\tilde{H}^{A, \text{sm}}$. \blacklozenge

We will consider preferably the basis $((P^{\bullet_q})^i)$ because, as shown by the previous proposition, it provides a basis of global sections of the small A-model D-module. We first compute the pairing $\tilde{S}^{A, \text{sm}}(\cdot, \cdot)$ in this basis.

Proposition 3.3.5. *The pairing $\tilde{S}^{A,\text{sm}}(\cdot, \cdot)$ in the basis $((P^{\bullet q})^i)$ is*

$$\tilde{S}^{A,\text{sm}}((P^{\bullet q})^i, (P^{\bullet q})^j) = \begin{cases} z^n m_1^{-1} & \text{if } i + j = n \\ z^n q w^{-w} m_1^{-1} & \text{if } i + j = n + \mu \\ 0 & \text{otherwise} \end{cases}$$

where $w^w := \prod_{i=0}^n w_i^{w_i}$.

Proof. Recall that $\tilde{S}^{A,\text{sm}}(\cdot, \cdot) := z^n \langle \cdot, \cdot \rangle$. We will use the formulas (28) and (30). The first case follows from the equivalence between $i + j = n$ and $c_i = c_j = 0$. From [22, Proposition 6.1.(3)], we have that $i + j = n + \mu$ is equivalent to $c_i + c_j = 1$ and $r(i) + r(j) = d_i - 1$. We conclude using the fact that $s_i s_j = w^{-w} \prod_{k \notin S_{c_i}} w_k^{-1}$ if $c_i + c_j = 1$. \square

Remark 3.3.6. Notice that if $w_0 = \dots = w_n = 1$ the bases $((P^{\bullet q})^i)_{0 \leq i \leq n}$ and $(\mathbf{1}_{f_i} P^j)$ are equal and that the pairing does not depend on q . \blacklozenge

Put

$$A_\infty := \frac{1}{2} \text{Diag}(\deg^{\text{orb}} 1, \deg^{\text{orb}} P, \dots, \deg^{\text{orb}}(P^{\bullet q})^{\mu-1})$$

The following proposition completes the description of the small A -model D -module \mathcal{S}_w^A .

Proposition 3.3.7. (1) *The matrix of the connection $\tilde{\nabla}^{A,\text{sm}}$ in the basis $(\mathbf{1}_{f_i} P^j)$ is*

$$(31) \quad -\frac{1}{z} C(q) \frac{dq}{q} + \left(\frac{1}{z} \mu C(q) + A_\infty \right) \frac{dz}{z}$$

(2) *The matrix of the connection $\tilde{\nabla}^{A,\text{sm}}$ in the basis $((P^{\bullet q})^i)$ is*

$$\left(-\frac{C^\varphi(q)}{z} + R^\varphi \right) \frac{dq}{q} + \left(\frac{\mu}{z} C^\varphi(q) + A_\infty \right) \frac{dz}{z}$$

where $R^\varphi := \text{diag}(c_0, \dots, c_{\mu-1})$ and

$$C^\varphi(q) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & q/w^w \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

Proof. (1) Since $c_1(T\mathbb{P}(w)) = \mu P$ by [22, lemma 3.21], we have

$$\tilde{\Phi}^{\text{sm}} = (P_{\bullet q}) \frac{dq}{q}, \quad \tilde{\mathfrak{E}}^{\text{sm}} = \mu P \text{ and } \tilde{\Phi}^{\text{sm}}(\tilde{\mathfrak{E}}^{\text{sm}}) = \mu(P_{\bullet q}).$$

The proposition then follows from the definition of $\tilde{\nabla}^{A,\text{sm}}$ (see equation (25)).

(2) Follows now from a straightforward computation via the change of basis (30). \square

Remark 3.3.8. (1) We have also $R^\varphi := \mu^{-1}(\text{Diag}(0, \dots, \mu - 1) - A_\infty)$ (compare with Theorem 4.3.2).

(2) As we have seen in proposition 3.3.3, the cohomology class $\mathbf{1}_{f_i} P^j$ does not define a global section of the small A -model D -module, whereas $(P^{\bullet q})^i$ does. This explains the fact that the matrix $C(q)$ (resp. $C^\varphi(q)$) contains rational (resp. integer) powers of q .

(3) Another way to measure the difference between the bases $(\mathbf{1}_{f_i} P^j)$ and $(P^{\bullet q})^i$ is to consider the restriction ∇ of $\tilde{\nabla}^{\text{sm}}$ to $\{\infty\} \times \mathcal{M}_A$. We have :

- $\nabla(\mathbf{1}_{f_i} P^j) = 0$,
- $\nabla((P^{\bullet q})^i) = R^\varphi((P^{\bullet q})^i) \frac{dq}{q}$.

In other words, the basis $(\mathbf{1}_{f_i} P^j)$ is ∇ -flat whereas $((P^{\bullet_q})^i)$ is not. \blacklozenge

Remark 3.3.9. The matrix $C(0)$ is the matrix of the endomorphism $P \cup_{\text{orb}}$ and does not generate the orbifold cohomology ring in general: from the matrix $C(0)$, we can not get all the orbifold product $\mathbf{1}_{f_i} P^j \cup_{\text{orb}} \mathbf{1}_{f_k} P^\ell$. \blacklozenge

Example 3.3.10. For $\mathbb{P}(1, 2, 2)$ we have

$$C(q) = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{4}q^{1/2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In particular,

$$C(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and we can not get the equality $\mathbf{1}_{1/2} \cup_{\text{orb}} \mathbf{1}_{1/2} P = P^2$ (see example 6.2.2 below) from $C(0)$.

4. B-MODEL

4.1. The setting. In [20], Iritani explains how to construct a mirror candidate from a toric stack. In the case of the weighted projective space $\mathbb{P}(1, w_1, \dots, w_n)$, we start with the following exact sequence

$$0 \longrightarrow \text{Pic}(\mathbb{P}(w)) \longrightarrow \mathbb{Z}^{n+1} \xrightarrow{\beta} N \longrightarrow 0$$

where $\beta : \mathbb{Z}^{n+1} \rightarrow N$ is the map defined via the stacky fan (see section 3.3.1). Applying the functor $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)$, we get :

$$1 \longrightarrow (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^{n+1} \xrightarrow{\pi} \mathbb{C}^* \longrightarrow 1$$

This gives our mirror candidate to $\mathbb{P}(w)$,

$$\begin{array}{ccc} (\mathbb{C}^*)^{n+1} & \xrightarrow{\tilde{F}} & \mathbb{C} \\ \pi \downarrow & & \\ \mathcal{M}_B & := & \mathbb{C}^* \end{array}$$

where $\tilde{F}(u_0, \dots, u_n) = \sum_{i=0}^n u_i$ and $\pi(u_0, \dots, u_n) = u_0 u_1^{w_1} \dots u_n^{w_n}$. Denote by x the coordinate on \mathcal{M}_B . As all the fibers of π are isomorphic to the torus $U := (\mathbb{C}^*)^n$, we can also consider

$$F : U \times \mathcal{M}_B \longrightarrow \mathbb{C}$$

defined by

$$(32) \quad F(u_1, \dots, u_n, x) = u_1 + \dots + u_n + \frac{x}{u_1^{w_1} \dots u_n^{w_n}}.$$

which is a deformation of $f : U \rightarrow \mathbb{C}$ defined by

$$f(u_1, \dots, u_n) = u_1 + \dots + u_n + \frac{1}{u_1^{w_1} \dots u_n^{w_n}}.$$

We will write

$$u_0 = \frac{1}{u_1^{w_1} \dots u_n^{w_n}}.$$

Remark 4.1.1. If we identify the monomial $\prod_{i=0}^n u_i^{a_i}$ with the point $(a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$, we see that each monomial u_i corresponds to the point $\beta(e_i) \in N$ where e_i is the canonical basis of \mathbb{Z}^{n+1} . We interpret $\beta(e_i)$ as the toric divisor D_i (see Remark 3.3.1). In particular, the monomial u_0 corresponds to $D_0 = \mathcal{O}(1)$ and we can expect that the multiplication by u_0 corresponds to the multiplication by $P := c_1(\mathcal{O}(1))$: this will be shown in section 5. ♦

4.2. Gauss-Manin systems and Brieskorn lattices. Let

$$G = \frac{\Omega^n(U)[x, x^{-1}, \tau, \tau^{-1}]}{(d_u - \tau d_u F) \wedge \Omega^{n-1}(U)[x, x^{-1}, \tau, \tau^{-1}]}$$

be the (Fourier-Laplace transform of the) Gauss-Manin system of F , and

$$G_0 = \frac{\Omega^n(U)[x, x^{-1}, \tau^{-1}]}{(\tau^{-1} d_u - d_u F) \wedge \Omega^{n-1}(U)[x, x^{-1}, \tau^{-1}]}$$

be (the Fourier-Laplace transform of) its Brieskorn lattice, where the notation d_u means that the differential is taken with respect to u only. G is equipped with a flat connection ∇^B defined by

$$\nabla_{\partial_\tau}^B(\omega_i \tau^i) = i\omega_i \tau^{i-1} - F\omega_i \tau^i \text{ and } \nabla_{\partial_x}^B(\omega_i \tau^i) = \mathcal{L}_{\partial_x}(\omega_i) \tau^i - \frac{\partial F}{\partial x} \omega_i \tau^{i+1}.$$

The Gauss-Manin system of f and its Brieskorn lattice are respectively defined by

$$G^o = \frac{\Omega^n(U)[\tau, \tau^{-1}]}{(d - \tau df) \wedge \Omega^{n-1}(U)[\tau, \tau^{-1}]}$$

and

$$G_0^o = \frac{\Omega^n(U)[\tau^{-1}]}{(\tau^{-1} d - df) \wedge \Omega^{n-1}(U)[\tau^{-1}]}.$$

G^o is also equipped with a flat connection $\nabla^{B,o}$ defined by

$$\nabla_{\partial_\tau}^{B,o}(\omega_i \tau^i) = i\omega_i \tau^{i-1} - f\omega_i \tau^i$$

(see for instance [13, Section 2]).

4.3. The canonical Saito structure. We look for a canonical trivial bundle on $\mathbb{P}^1 \times \mathcal{M}_B$, equipped with a connection and a flat pairing as in section 3. A canonical solution of the Birkhoff problem for the Brieskorn lattice G_0 (see theorem 4.3.2 below) yields such objects.

4.3.1. The canonical trivial bundle. Let

$$\Gamma_0 = \{(y_1, \dots, y_n) \in \mathbb{R}^n | y_1 + \dots + y_n = 1\}$$

and

$$\chi_{\Gamma_0} = u_1 \frac{\partial}{\partial u_1} + \dots + u_n \frac{\partial}{\partial u_n},$$

$$\Gamma_j = \left\{ (y_1, \dots, y_n) \in \mathbb{R}^n | y_1 + \dots + y_{j-1} + \left(1 - \frac{\mu}{w_j}\right) y_j + \dots + y_n = 1 \right\}$$

and

$$\chi_{\Gamma_j} = u_1 \frac{\partial}{\partial u_1} + \dots + u_{j-1} \frac{\partial}{\partial u_{j-1}} + \left(1 - \frac{\mu}{w_j}\right) u_j \frac{\partial}{\partial u_j} + \dots + u_n \frac{\partial}{\partial u_n}$$

for $j = 1, \dots, n$. We define, for $j = 0, \dots, n$,

$$h_{\Gamma_j} = \chi_{\Gamma_j}(F) - F.$$

We thus have $h_{\Gamma_0} = -\mu x u_0$ and $h_{\Gamma_j} = -\frac{\mu}{w_j} u_j$ if $j = 1, \dots, n$. Last we put, for $g = u_1^{r_1} \dots u_n^{r_n}$,

$$\phi_{\Gamma_0}(g) = r_1 + \dots + r_n$$

and, for $j = 1, \dots, n$,

$$\phi_{\Gamma_j}(g) = r_1 \cdots + r_{j-1} + \left(1 - \frac{\mu}{w_j}\right) r_j + \cdots + r_n.$$

We will write ∂_τ instead of $\nabla_{\partial_\tau}^B$ for short.

Lemma 4.3.1. *Let ω_0 be the class of $\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}$ in G . One has, for any monomial g , the equality*

$$(\tau \partial_\tau + \phi_{\Gamma_j}(g))g\omega_0 = \tau h_{\Gamma_j} g\omega_0$$

in G , where $g\omega_0$ denotes the class of $g \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}$ in G . In particular, $\tau \partial_\tau \omega_0 = \tau h_{\Gamma_0} \omega_0$.

Proof. Straightforward. \square

This lemma is the starting point in order to solve the Birkhoff problem for G_0 , as it has been the starting point to solve the one for G_0° in [14, section 3]. Put $\omega_0^\varphi = \omega_0$ and $\omega_1^\varphi = xu_0\omega_0$: the equality

$$\tau \partial_\tau \omega_0 = \tau h_{\Gamma_0} \omega_0$$

becomes

$$-\frac{1}{\mu} \tau \partial_\tau \omega_0^\varphi = \tau \omega_1^\varphi.$$

Iterating the process (the idea is to define $\omega_2^\varphi = -\frac{1}{\mu} \omega_1^\varphi h_{\Gamma_1}$ etc...), one gets sections $\omega_1^\varphi, \dots, \omega_{\mu-1}^\varphi$ of G satisfying

$$-\frac{1}{\mu} (\tau \partial_\tau + \alpha_k) \omega_k^\varphi = \tau \omega_{k+1}^\varphi$$

for $k = 1, \dots, \mu - 2$ and

$$-\frac{1}{\mu} (\tau \partial_\tau + \alpha_{\mu-1}) \omega_{\mu-1}^\varphi = \frac{x}{w_1^{w_1} \cdots w_n^{w_n}} \tau \omega_0^\varphi.$$

This can be done as in [14, section 2 and proof of proposition 3.2].

By construction we have

$$\omega_k^\varphi = \frac{x}{w_1^{a(k)_1} \cdots w_n^{a(k)_n}} u_0 u_1^{a(k)_1} \cdots u_n^{a(k)_n} \omega_0^\varphi$$

for $k = 1, \dots, \mu - 1$ where the multi-indices $a(k) = (a(k)_0, a(k)_1, \dots, a(k)_n) \in \mathbb{N}^{n+1}$ are defined in section 2.1 (notice that $a(k)_0 = 1$ for $k \geq 1$ because $w_0 = 1$). We will put $u^{a(k)} = u_0 u_1^{a(k)_1} \cdots u_n^{a(k)_n}$: for instance, $u^{a(1)} = u_0$ and $u^{a(\mu)} = 1$ because u_0 is defined by the equation $u_0 u_1^{w_1} \cdots u_n^{w_n} = 1$.

Recall the rational numbers α_k defined in section 2.1. Let

$$A_\infty = \text{Diag}(\alpha_0, \dots, \alpha_{\mu-1}),$$

and, for $x \in \mathcal{M}_B$,

$$A_0^\varphi(x) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \mu x / w^w \\ \mu & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdot & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdots & \mu & 0 \end{pmatrix}$$

where $w^w = w_1^{w_1} \cdots w_n^{w_n}$. We will preferably express our results in the variable $\theta := \tau^{-1}$, also denoted on the A-side by z .

Theorem 4.3.2. *The classes $\omega_0^\varphi, \dots, \omega_{\mu-1}^\varphi$ form a basis ω^φ of G_0 over $\mathbb{C}[x, x^{-1}, \theta]$. In this basis, the matrix of the connection ∇^B is*

$$\left(\frac{A_0^\varphi(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} + \left(-\frac{A_0^\varphi(x)}{\theta} - A_\infty + H \right) \frac{dx}{\mu x}$$

where $H = \text{Diag}(0, 1, \dots, \mu - 1)$.

Proof. One shows that G_0 is finitely generated as in [14, proposition 3.2], with the help of lemma 4.3.1. To show that it is free notice that, again by [14, proposition 3.2], a section of the kernel of the surjective map

$$(\mathbb{C}[x, x^{-1}])^\mu \rightarrow G_0 \rightarrow 0$$

is given by μ Laurent polynomials which vanishes everywhere (see remark below). This gives the first assertion. Let us show the second one: the assertion about $\nabla_{\partial_\theta}^B$ is clear, thanks to the definition of the ω_k 's. The action of $\nabla_{\partial_x}^B$ is defined, for $\eta \in G_0$, by

$$\nabla_{\partial_x}^B(\eta) = -u_0\eta\theta^{-1} + \mathcal{L}_{\partial_x}(\eta)$$

and we have, for $\eta = u_0 u_1^{r_1} \dots u_n^{r_n} \omega_0$,

$$u_0\eta = \frac{1}{\mu x} F\eta - \frac{1}{\mu x} \theta \left(\sum_{i=1}^n r_i - w_i \right) \eta.$$

We deduce from this, because $\theta^2 \nabla_{\partial_\theta}^B$ is induced by the multiplication by F , that

$$\nabla_{\partial_x}^B \omega_k^\varphi = -\frac{A_0(x)}{\mu x} \theta^{-1} (\omega_k^\varphi) + \frac{1}{\mu x} \left(\mu + \sum_{i=1}^n a(k)_i - \sum_{i=1}^n w_i - \alpha_k \right) \omega_k^\varphi.$$

Now, one has $\sum_{i=1}^n a(k)_i = k - 1$ (see section 2.1) and $\sum_{i=1}^n w_i = \mu - 1$ so that

$$\mu + \sum_{i=1}^n a(k)_i - \sum_{i=1}^n w_i - \alpha_k = k - \alpha_k.$$

□

Remark 4.3.3. (1) Let $x \in \mathcal{M}_B$. The previous construction gives the canonical solution $\omega^o = (\omega_0^o, \dots, \omega_{\mu-1}^o)$ of the Birkhoff problem for the Brieskorn lattice of $F_x := F(\cdot, x)$, obtained using the methods in [14].

(2) The deformation F can be seen as a 'rescaling' of the function f and it is possible to present the proof of the previous proposition in a slightly different way. However, we prefer to keep our more direct approach because it emphasizes the multiplication by u_0 (see the last part of section 4.1) and gives the general way to proceed if one wants to compute other examples, e.g $F(u_1, u_2, x) = u_1 + u_2 + \frac{1}{u_1 u_2^2} + \frac{x}{u_2}$.

(3) In order to make the link with the J -function and quantum differential operators, notice that

$$[w^w \theta^\mu \prod_{i=1}^\mu (x \nabla_{\partial_x} - c_i) - x] \omega_0^\varphi = 0$$

(compare with [8, corollary 1.8]). ♦

Remark 4.3.4. (Various generalizations)

(1) The case $w_0 \neq 1$ can be handled using the presentation of the Gauss-Manin system considered in [10]. This is longer but yields the same result: one has to replace $w_1^{a(k)_1} \dots w_n^{a(k)_n}$ by $w_0^{a(k)_0} w_1^{a(k)_1} \dots w_n^{a(k)_n}$ in the definition of the ω_k^φ 's and $w_1^{w_1} \dots w_n^{w_n}$ by $w_0^{w_0} w_1^{w_1} \dots w_n^{w_n}$ in

the definition of $A_0^\varphi(x)$.

(2) One could start more generally with the function

$$f(u_1, \dots, u_n) = b_1 u_1 + \dots + b_n u_n + \frac{1}{u_1^{w_1} \dots u_n^{w_n}}$$

where b_1, \dots, b_n are complex numbers such that $b_1 \dots b_n \neq 0$ and would obtain analogous results. The Laurent polynomial considered in [14] is obtained putting $b_i = w_i$ for all i in f . But, if we keep in mind mirror symmetry, only the case $b_i = 1$ will be really relevant (see theorem 5.1.1 and section 4.1). ♦

Put $R^\varphi = \mu^{-1}(H - A_\infty)$. It follows from section 2.1 that

$$R^\varphi = \mu^{-1}(H - A_\infty) = \text{Diag}(c_0, \dots, c_{\mu-1})$$

and from theorem 4.3.2 that the matrix of $x\nabla_{\partial_x}^B$ in the basis ω^φ is given by

$$-\mu^{-1} \frac{A_0^\varphi(x)}{\theta} + R^\varphi.$$

Let \mathcal{L}^φ be the $\mathbb{C}[x, \theta, \theta^{-1}]$ -submodule of G generated by ω^φ : $x\nabla_{\partial_x}^B$ induces a map on $\mathcal{L}^\varphi/x\mathcal{L}^\varphi$ whose eigenvalues are contained in $[0, 1[$, because $A_0^\varphi(0)$ is a Jordan matrix and because $c_k \in [0, 1[$ for $k = 0, \dots, \mu - 1$. Thus we get

Corollary 4.3.5. *The lattice \mathcal{L}^φ is Deligne's canonical extension of the Gauss-Manin system G to $\mathbb{C}^* \times \mathbb{C}$ such that the eigenvalues of the residue of $\nabla_{\partial_x}^B$ are contained in $[0, 1[$. □*

The index φ recalls the "vanishing cycles". We will call the basis ω^φ the *canonical basis*, as it is suggested by this corollary and remark 4.3.3 (1).

Theorem 4.3.2 says that the canonical basis ω^φ gives an extension of G_0 as a trivial bundle H^B on $\mathbb{P}^1 \times \mathcal{M}_B$ (the module of its global sections is generated by $\omega_0^\varphi, \dots, \omega_{\mu-1}^\varphi$) equipped with a connection ∇^B with logarithmic pole at $\tau := \theta^{-1} = 0$ and pole of Poincaré rank less or equal to one at $\theta = 0$ (see for instance [25, section 2.1]). The following definition is thus consistent:

Definition 4.3.6. The bundle H^B is the *canonical trivial bundle*.

4.3.2. The flat and the orbifold bases. Let Δ be an open disc in \mathbb{C}^* and, for $x \in \Delta$, $\omega^{\text{flat}} := \omega^\varphi x^{-R^\varphi}$. ω^{flat} is a local basis of G_0^{an} and we will call it *the flat basis*, flat with respect to the restriction ∇ of ∇^B at $\{\theta = \infty\} \times \mathbb{C}^*$. The matrix of the connection ∇^B in the basis ω^{flat} is

$$\left(\frac{A_0^{\text{flat}}(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} - \frac{A_0^{\text{flat}}(x)}{\theta} \frac{dx}{\mu x}$$

where

$$A_0^{\text{flat}}(x) = \mu \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & x^{1-c_{\mu-1}}/w^w \\ x^{c_1-c_0} & 0 & 0 & \dots & 0 & 0 \\ 0 & x^{c_2-c_1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & x^{c_{\mu-1}-c_{\mu-2}} & 0 \end{pmatrix},$$

the c_i 's being defined in section 2.1.

For $i \in \{0, \dots, \mu - 1\}$, we denote

$$(33) \quad \omega_i^{\text{orb}} := s_i^{-1} \omega_i^{\text{flat}} = x^{-c_i} s_i^{-1} \omega_i^\varphi$$

where the s_i are defined in (30). The matrix of the connection ∇^B in the basis ω^{orb} is

$$\left(\frac{A_0^{\text{orb}}(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} - \frac{A_0^{\text{orb}}(x)}{\theta} \frac{dx}{\mu x}$$

where

$$A_0^{\text{orb}}(x) = \mu \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_\mu x^{1-c_{\mu-1}}/w^w \\ a_1 x^{c_1-c_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 x^{c_2-c_1} & 0 & \cdots & 0 & 0 \\ \ddots & \cdots & \cdot & \cdots & \cdot & \cdot \\ \ddots & \cdots & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdots & a_{\mu-1} x^{c_{\mu-1}-c_{\mu-2}} & 0 \end{pmatrix},$$

the a_i 's being defined in (29).

4.4. The pairing. We define in this section a nondegenerate, symmetric and ∇^B -flat bilinear form on G_0 . The lattice G_0^o is equipped with a nondegenerate bilinear form

$$S^o : G_0^o \times G_0^o \rightarrow \mathbb{C}[\theta]\theta^n,$$

$\nabla^{B,o}$ -flat and satisfying , for $p(\theta) \in \mathbb{C}[\theta]$,

$$p(\theta)S^o(\cdot, \cdot) = S^o(p(\theta)\cdot, \cdot) = S^o(\cdot, p(-\theta)\cdot).$$

More precisely, in the basis $\omega^o = (\omega_0^o, \dots, \omega_{\mu-1}^o)$ of G_0^o considered in remark 4.3.3 (1), one has

$$S^o(\omega_k^o, \omega_\ell^o) = \begin{cases} S^o(\omega_0^o, \omega_n^o) \in \mathbb{C}^* \theta^n & \text{if } 0 \leq k \leq n \text{ and } k + \ell = n, \\ w^{-w} S^o(\omega_0^o, \omega_n^o) & \text{if } n+1 \leq k \leq \mu-1 \text{ and } k + \ell = \mu + n, \\ 0 & \text{otherwise} \end{cases}$$

where $w^w = w_1^{w_1} \cdots w_n^{w_n}$ as above. This is shown as in [14, Sect. 4]. From now on, we will choose the normalization $S^o(\omega_0^o, \omega_n^o) = 1/m_1 \theta^n$ (recall that $m_1 = w_1 \cdots w_n$).

We define, in the basis ω^φ given by theorem 4.3.2,

$$(34) \quad S^B(\omega_k^\varphi, \omega_\ell^\varphi) = \begin{cases} m_1^{-1} \theta^n & \text{if } 0 \leq k \leq n \text{ and } k + \ell = n, \\ w^{-w} \frac{x}{m_1} \theta^n & \text{if } n+1 \leq k \leq \mu-1 \text{ and } k + \ell = \mu + n, \\ 0 & \text{otherwise} \end{cases}$$

This gives

$$S^B : G_0 \times G_0 \rightarrow \mathbb{C}[x, x^{-1}, \theta] \theta^n$$

by linearity, using the rules

$$a(x, \theta)S(\cdot, \cdot) = S(a(x, \theta)\cdot, \cdot) = S(\cdot, a(x, -\theta)\cdot)$$

for $a(x, \theta) \in \mathbb{C}[x, \theta]$. Flatness is defined by equations (16), (17) (replacing z by θ and ∂_k by ∂_x). The following lemma justifies the definition of S^B :

Lemma 4.4.1. *The bilinear form S^B is ∇^B -flat.*

Proof. We work in the basis ω^φ : it follows first from the definition of $A_0^\varphi(x)$ and S^B that one has $(A_0^\varphi(x))^* = A_0^\varphi(x)$ where $*$ denotes the adjoint with respect to S^B . The symmetry property of the numbers α_k (see corollary 2.1.2) shows also that $A_\infty + A_\infty^* = nI$. This gives equation (16). Now, equation (17) reads

$$x \partial_x S^B(\omega_i^\varphi, \omega_j^\varphi) = S^B(R^\varphi(\omega_i^\varphi), \omega_j^\varphi) + S^B(\omega_i^\varphi, R^\varphi(\omega_j^\varphi))$$

but this follows once again from lemma 2.1.2. \square

Corollary 4.4.2. *We have*

$$S^B(\omega_k^{\text{orb}}, \omega_\ell^{\text{orb}}) = \begin{cases} m_1^{-1} \theta^n & \text{if } 0 \leq k \leq n \text{ and } k + \ell = n, \\ m_{i+1}^{-1} \theta^n & \text{if } d_1 + \cdots + d_i \leq k < d_1 + \cdots + d_{i+1} \text{ and } k + \ell = \mu + n, \\ 0 & \text{otherwise} \end{cases}$$

Proof. By lemma 4.4.1, S^B is constant in the basis ω^{flat} thus in the basis ω^{orb} and the result follows from the definitions, using the fact that $m_i = m_j$ if $i + j = k + 2$ and $m_1 \cdots m_k = w^w$. \square

Remark 4.4.3. (1) The coefficient of θ^n in $S^B(\varepsilon, \eta)$, $\varepsilon, \eta \in G_0$, depends only on the classes of ε and η in $G_0/\theta G_0$. We will denote it by $g([\varepsilon], [\eta])$. This defines a nondegenerate bilinear form g on $G_0/\theta G_0$, see [24, p. 211].

(2) The bilinear form S^B defines a bilinear form (also denoted by S^B) on the trivial bundle H^B (see for instance [25, section 1.4]). \blacklozenge

4.5. Résumé (the canonical Saito structure). We have constructed a canonical trivial bundle H^B (see definition 4.3.6), equipped with a flat meromorphic connection ∇^B , and a ∇^B -flat pairing S^B (see section 4.4). Summarizing, we get

Theorem 4.5.1. *The tuple*

$$\mathcal{S}_w^B = (\mathcal{M}_B, H^B, \nabla^B, S^B, n)$$

is a canonical Saito structure.

It should be emphasized that we have something 'canonical' here.

Definition 4.5.2. \mathcal{S}_w^B is the *small B-model D-module*.

Remark 4.5.3. Iritani's B -model D -module (see [20, definition 3.16]) is *a priori* different (compare with remark 3.1.6), as he deals only with bundles on $\mathbb{C} \times \mathcal{M}_B$: in particular, he doesn't consider the Birkhoff problem at all. \blacklozenge

5. THE MIRROR PARTNER OF THE SMALL QUANTUM ORBIFOLD COHOMOLOGY OF $\mathbb{P}(w)$

5.1. Correspondance. Let us summarize the results obtained. On the both sides we have a trivial bundle over a base isomorphic to $\mathbb{P}^1 \times \mathbb{C}^*$. The free $\mathbb{C}[q, q^{-1}]$ -module H_A of global sections of $\tilde{H}^{A, \text{sm}}$ is generated by $(P^\bullet)^j$ for $j = 0, \dots, \mu - 1$ whereas the free $\mathbb{C}[x, x^{-1}]$ -module H_B of global sections of H^B is generated by (ω_i^φ) . Define

$$\gamma : H_A \rightarrow H_B$$

by

$$\gamma(P^\bullet{}^j) = \omega_j^\varphi.$$

This gives an isomorphism between H_A and H_B , after identifying $\mathbb{P}^1 \times \mathcal{M}_A$ and $\mathbb{P}^1 \times \mathcal{M}_B$ via the map $(z, q) \mapsto (\theta, x)$. The following theorem should be compared with [20, Proposition 4.8].

Theorem 5.1.1. *The map γ yields an isomorphism between the small A-model D-module*

$$(\mathcal{M}_A, \tilde{H}^{A, \text{sm}}, \tilde{\nabla}^{A, \text{sm}}, \tilde{S}^{A, \text{sm}}, n)$$

and the small B-model D-module

$$(\mathcal{M}_B, H^B, \nabla^B, S^B, n).$$

Proof of Theorem 5.1.1. We first show that the matrices of the connections in the bases $(P^\bullet{}^j)$ and (ω_i^φ) are the same. We have $\deg^{\text{orb}}(\mathbf{1}_{f_1}) = 0 = \alpha_0$ and

$$\deg^{\text{orb}}(\mathbf{1}_{f_i}) = 2(d_1 + \cdots + d_{i-1} - \mu f_i) = 2\alpha_{d_1 + \cdots + d_{i-1}}$$

if $i \geq 2$. Thus,

$$\deg^{\text{orb}}(\mathbf{1}_{f_i} P^j) = 2(d_1 + \cdots + d_{i-1} - \mu f_i) + 2j = 2\alpha_{d_1 + \cdots + d_{i-1} + j}.$$

Finally, $\deg^{\text{orb}}(P^{\bullet j}) = 2\alpha_j$ and this shows that the matrices A_∞ are the same. The remaining assertions are clear by sections 4.3.2 and 3.3.2. For the pairing, it is enough to notice that

$$\widetilde{S}^{A,\text{sm}}(P^{\bullet i}, P^{\bullet j}) = S^B(\gamma(P^{\bullet i}), \gamma(P^{\bullet j}))$$

but this follows from the formula (34) and proposition 3.3.5. \square

We can thus identify the A -model D -module \mathcal{S}_w^A and the B -model D -module \mathcal{S}_w^B : the result is a canonical Saito structure and will be denoted by

$$\mathcal{S}_w := (\mathcal{M}, H, \nabla, S, n).$$

We also get, with the help of proposition 2.2.5, a canonical Frobenius type structure

$$\mathbb{F}_w = (\mathcal{M}, E, \nabla, R_0, R_\infty, \Phi, g)$$

on \mathcal{M} where $E := G_0/\theta G_0 = \Omega^n(U)/d_u F \wedge \Omega^{n-1}(U)$.

Definition 5.1.2. (1) The tuple \mathcal{S}_w is called the w -Saito structure.

(2) The tuple \mathbb{F}_w is called the w -Frobenius type structure.

These objects will give canonical Frobenius manifolds.

5.2. The small quantum product via the Jacobi algebra. We give here a mirror partner of the small quantum product. This will give an interpretation of the products $P^{\bullet i} \bullet_q P^{\bullet j}$ in terms of commutative algebra.

For $k = 0, \dots, \mu - 1$, put $\omega_k^\varphi = h_k^\varphi \omega_0^\varphi$ where $h_0^\varphi = 1$ and

$$h_k^\varphi = \frac{x}{w^{a(k)}} u^{a(k)}$$

for $k = 1, \dots, \mu - 1$ (see section 4.3). We define now the product $*$ on E by

$$[\omega_i^\varphi] *_x [\omega_j^\varphi] := [h_i^\varphi h_j^\varphi \omega_0^\varphi].$$

Proposition 5.2.1. Let $i, j \in \{0, \dots, \mu - 1\}$. If $i + j \geq \mu$, we denote $\overline{i+j} := i + j - \mu$.

(1) We have, in E ,

$$(35) \quad [\omega_i^\varphi] *_x [\omega_j^\varphi] = \begin{cases} [\omega_{i+j}^\varphi] & \text{if } i + j \leq \mu - 1, \\ xw^{-w}[\omega_{\overline{i+j}}^\varphi] & \text{if } i + j \geq \mu \end{cases}$$

In particular, $[\omega_i^\varphi] = [\omega_1^\varphi]^{*i} := \underbrace{[\omega_1^\varphi] *_x \cdots *_x [\omega_1^\varphi]}_{i \text{ times}}$.

(2) We have, in $H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$,

$$(36) \quad P^{\bullet i} \bullet_q P^{\bullet j} = \begin{cases} P^{\bullet(i+j)} & \text{if } i + j \leq \mu - 1, \\ qw^{-w} P^{\bullet(\overline{i+j})} & \text{if } i + j \geq \mu \end{cases}$$

Proof. (1) Because $u_0 u_1^{w_1} \cdots u_n^{w_n} = 1$ and, for $i \geq 1$, $u^{a(i)} w^{-a(i)} \omega_0^\varphi = x^{i-1} u_0^i \omega_0^\varphi$ in E . (2) Follows from proposition 3.3.7. \square

Corollary 5.2.2. The matrix $\frac{1}{\mu} A_0^\varphi(x)$ in theorem 4.3.2 represents the endomorphism $[\omega_1^\varphi] *_x$ in the basis $[\omega^\varphi]$.

At the end, we get the announced relationship:

Corollary 5.2.3. The product $*_x$ is the mirror partner of the small quantum product \bullet_q : we have

$$[\gamma(P^{\bullet i})] *_x [\gamma(P^{\bullet j})] = [\gamma(P^{\bullet i} \bullet_q P^{\bullet j})].$$

Proof. Follows from proposition 5.2.1 and the definition of γ . \square

6. LIMITS

Up to now, we have worked on $\mathcal{M} = \mathbb{C}^*$ and we want now to define a limit at 0 of the structure \mathcal{S}_w (*resp.* \mathbb{F}_w). This should be of course a Saito structure (*resp.* a Frobenius type structure) on a \mathbb{P}^1 (*resp.* on a point), as canonical as possible. This canonical limit will be constructed with the help of the Kashiwara-Malgrange V -filtration at the origin ⁶. The canonical limit Frobenius type structure (on a point) will be then obtain, using the proposition 2.2.5.

Notice that, setting “ $q = 0$ ” on the A-side, one expects to get the orbifold cohomology, the endomorphism $c_1(T\mathbb{P}(w)) \cup_{\text{orb}}$ and the orbifold Poincaré pairing. We will see that this guess agrees with our result.

6.1. Canonical limits of the structures \mathcal{S}_w and \mathbb{F}_w . We apply the receipe announced above. For convenience reasons, we start from the B -model and we use the notations of section 4, forgetting the index B .

6.1.1. The V -filtration at $x = 0$. Recall the basis $\omega^\varphi = (\omega_0^\varphi, \dots, \omega_{\mu-1}^\varphi)$ of G_0 over $\mathbb{C}[x, x^{-1}, \theta]$, which is also a basis of G over $\mathbb{C}[x, x^{-1}, \theta, \theta^{-1}]$. Put $v(\omega_0^\varphi) = \dots = v(\omega_n^\varphi) = 0$ and, for $k = n+1, \dots, \mu-1$, $v(\omega_k^\varphi) = c_k$. Define, for $0 \leq \alpha < 1$,

$$\begin{aligned} V^\alpha G &= \sum_{\alpha \leq v(\omega_k^\varphi)} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^\varphi + x \sum_{\alpha > v(\omega_k^\varphi)} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^\varphi, \\ V^{>\alpha} G &= \sum_{\alpha < v(\omega_k^\varphi)} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^\varphi + x \sum_{\alpha \geq v(\omega_k^\varphi)} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^\varphi \end{aligned}$$

and $V^{\alpha+p}G = x^p V^\alpha G$ for $p \in \mathbb{Z}$ and $\alpha \in [0, 1[$. This gives a decreasing filtration V^\bullet of G by $\mathbb{C}[x][\theta, \theta^{-1}]$ -submodules such that

$$V^\alpha G = \mathbb{C}[\theta, \theta^{-1}] \langle \omega_k^\varphi | v(\omega_k^\varphi) = \alpha \rangle + V^{>\alpha} G.$$

Notice that the canonical lattice \mathcal{L}^φ (see section 4.3.1) is equal to $V^0 G$ and that $\mathcal{L}^\varphi / x \mathcal{L}^\varphi = V^0 G / V^1 G$. We will put $G^\alpha := V^\alpha G / V^{>\alpha} G$ and $\overline{G} := \bigoplus_{\alpha \in [0, 1[} G^\alpha$.

Lemma 6.1.1. (1) For each α , $(x \nabla_{\partial_x} - \alpha)$ is nilpotent on G^α .

(2) Let N be the nilpotent endomorphism of \overline{G} which restricts to $(x \nabla_{\partial_x} - \alpha)$ on G^α . Its Jordan blocks are in one to one correspondance with the maximal constant sequences in $(c_0, \dots, c_{\mu-1})$ and the corresponding sizes are the same.

(3) The classes $[\omega_0^\varphi], \dots, [\omega_{\mu-1}^\varphi]$ give a basis $[\omega^\varphi]$ of \overline{G} over $\mathbb{C}[\theta, \theta^{-1}]$.

Proof. (1) It suffices to prove the assertion for $\alpha \in [0, 1[$. It follows from theorem 4.3.2 that we have

$$x \nabla_{\partial_x} \omega_k^\varphi = -\frac{1}{\theta} \omega_{k+1}^\varphi$$

for $k = 0, \dots, n-1$ and $x \nabla_{\partial_x} \omega_n^\varphi \in V^{>0} G$. Moreover we have, for $k = n+1, \dots, \mu-2$,

$$(x \nabla_{\partial_x} - c_k) \omega_k^\varphi = -\frac{1}{\theta} \omega_{k+1}^\varphi$$

and this is equal to 0 in $G^{v(\omega_k^\varphi)}$ if $c_{k+1} > c_k$. Last,

$$(x \nabla_{\partial_x} - c_{\mu-1}) \omega_{\mu-1}^\varphi = -\frac{1}{\theta} x w^{-w} \omega_0^\varphi \in x \sum_{v(\omega_{\mu-1}^\varphi) \geq v(\omega_k^\varphi)} \mathbb{C}[x] \omega_k^\varphi \subset V^{>c_{\mu-1}} G.$$

(2) follows from (1) and (3) follows from the definition of V^\bullet . \square

⁶Naively, one could set $q = x = 0$ in the matrices of ∇ and S in the flat basis. Unfortunately, these matrices are multivalued (they have rational power of q, x), so that this limit process does not make sense.

The matrix of N in the basis $[\omega^\varphi]$ is $B\theta^{-1}$ where $B_{i,j} = 0$ if $i \neq j+1$, $B_{i+1,i} = -1$ if $c_i = c_{i-1}$ and $B_{i+1,i} = 0$ if $c_i \neq c_{i-1}$ (notice that $-\mu B = A_0^{\text{flat}}(0)$).

Corollary 6.1.2. *The filtration V^\bullet is the Kashiwara-Malgrange filtration at $x = 0$.*

Proof. By the previous lemma, the filtration V^\bullet satisfies all the characteristic properties of the Kashiwara-Malgrange filtration. \square

6.1.2. *Limits.* The free $\mathbb{C}[\theta, \theta^{-1}]$ -module \overline{G} is equipped with a connection $\overline{\nabla}$ whose matrix in the basis $[\omega^\varphi]$ is

$$\left(\frac{\overline{A}_0}{\theta} + A_\infty \right) \frac{d\theta}{\theta}$$

where $\overline{A}_0 = -\mu B$ and $A_\infty = \text{Diag}(\alpha_0, \dots, \alpha_{\mu-1})$. We now need a limit bilinear form. Let \overline{G}_0 be the $\mathbb{C}[\theta]$ -submodule of \overline{G} generated by $[\omega_0^\varphi], \dots, [\omega_{\mu-1}^\varphi]$ and define

$$\overline{S} : \overline{G}_0 \times \overline{G}_0 \rightarrow \mathbb{C}[\theta]\theta^n$$

by

$$\overline{S}([\omega_k^\varphi], [\omega_{n-k}^\varphi]) = \frac{1}{w_1 \cdots w_n} \theta^n$$

for $k = 0, \dots, n$ (in which case $c_k = c_{n-k} = 0$),

$$\overline{S}([\omega_k^\varphi], [\omega_{\mu+n-k}^\varphi]) = \frac{1}{w_1^{w_1+1} \cdots w_n^{w_n+1}} \theta^n$$

for $k = n+1, \dots, \mu-1$ (in which case $c_k + c_{\mu+n-k} = 1$) and $\overline{S}([\omega_i^\varphi], [\omega_j^\varphi]) = 0$ otherwise. The pairing \overline{S} is induced by S on \overline{G} (hence it is indeed a limit): this is shown as in [26, remark 3.6] (with only mild modifications) because

$$S(V^\beta G, V^{1-\beta} G) \subset x\mathbb{C}[x, \theta, \theta^{-1}]$$

if $\beta \neq 0$ (and thus the induced bilinear form on the graded pieces is obtained taking the coefficient of x) and

$$S(V^0 G, V^0 G) \subset \mathbb{C}[x, \theta, \theta^{-1}]$$

where V^\bullet is the Kashiwara-Malgrange filtration at $x = 0$ defined above.

As in section 4.3, we get an extension of \overline{G}_0 as a trivial bundle \overline{H} on \mathbb{P}^1 , equipped with a connection $\overline{\nabla}$ and a pairing \overline{S} .

Theorem 6.1.3. *The tuple $\overline{\mathcal{S}}_w = (\overline{H}, \overline{\nabla}, \overline{S}, n)$ is a Saito structure on \mathbb{P}^1 .*

Proof. It remains to show that \overline{S} is $\overline{\nabla}$ -flat, and it is enough to show that $(\overline{A}_0)^* = \overline{A}_0$ and $A_\infty + (A_\infty)^* = n \text{id}$. The second equality follows easily from lemma 2.1.1 and from the definition of \overline{S} . To show the first one, use moreover lemma 6.1.1, the key point being that $\overline{S}(\overline{A}_0([\omega_n^\varphi]), [\omega_j^\varphi]) = 0 = \overline{S}([\omega_n^\varphi], \overline{A}_0([\omega_j^\varphi]))$ because, by lemma 6.1.1, $\overline{A}_0([\omega_n^\varphi]) = 0$ and because $[\omega_0^\varphi]$ does not belong to the image of \overline{A}_0 . \square

Remark 6.1.4. It should be emphasized that the conclusion of the previous theorem is not always true if we work directly on $\mathcal{L}^\varphi/x\mathcal{L}^\varphi$, that is if we forget the gr^V , because the matrix $A_0^\varphi(0)$ is not 'enough symmetric'. \blacklozenge

Definition 6.1.5. The tuple $\overline{\mathcal{S}}_w$ is the canonical limit Saito structure.

Define now $\overline{E} = \overline{G}_0/\theta\overline{G}_0$ and let $[\omega^\varphi]$ be the basis of \overline{E} induced by $[\omega^\varphi]$. As explained in section 2.2, \overline{E} is thus equipped with two endomorphisms \overline{R}_0 and \overline{R}_∞ (with respective matrices \overline{A}_0 and $-A_\infty$) and with a nondegenerate bilinear form \overline{g} obtained from \overline{S} as in remark 4.4.3.

Corollary 6.1.6. *The tuple*

$$\overline{\mathbb{F}}_w = (\overline{E}, \overline{R}_0, \overline{R}_\infty, \overline{g})$$

is a Frobenius type structure on a point.

Definition 6.1.7. $\overline{\mathbb{F}}_w$ is the canonical limit Frobenius type structure.

Remark 6.1.8. Let (E, A, B, g) be a Frobenius type structure on a point. We will say that an element e of E is a *pre-primitive section* if $(e, A(e), \dots, A^{\mu-1}(e))$ is a basis of E over \mathbb{C} and that e is *homogeneous* if it is an eigenvector of B . Recall that $[\omega_0^\varphi]$ denotes the class of ω_0^φ in E . Then $[\omega_0^\varphi]$ is a pre-primitive and homogeneous section of the limit Frobenius type structure $(E, \overline{R}_0, \overline{R}_\infty, \overline{g})$ if and only if $\mu = n + 1$. If $\mu \geq n + 2$, this Frobenius type structure has no pre-primitive section at all. ♦

6.2. Application: the mirror partner of the orbifold cohomology ring. We define, on the graded vector space \overline{E} , a commutative and associative product \cup by (see proposition 5.2.1)

$$[\omega_i^\varphi] \cup [\omega_j^\varphi] := \frac{1}{w^w} [\omega_{i+j}^\varphi] \text{ if } i + j \geq \mu \text{ and } 1 + c_{i+j} = c_i + c_j,$$

$$[\omega_i^\varphi] \cup [\omega_j^\varphi] := [\omega_{i+j}^\varphi] \text{ if } i + j \leq \mu - 1 \text{ and } c_{i+j} = c_i + c_j$$

and $[\omega_i^\varphi] \cup [\omega_j^\varphi] = 0$ otherwise. This product is homogeneous and $[\omega_0^\varphi]$ is the unit. The bilinear form \overline{g} on \overline{E} is also homogeneous because $\overline{g}([\omega_i^\varphi], [\omega_j^\varphi]) \neq 0$ only if $i + j = n$ or if $i + j = \mu + n$: in any case, $\alpha_i + \alpha_j = n$.

Proposition 6.2.1. *The tuple $(\overline{E}, \cup, \overline{g})$ is a Frobenius algebra, isomorphic to*

$$(H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C}), \cup_{\text{orb}}, \langle \cdot, \cdot \rangle).$$

Proof. To prove the first assertion, it remains to show the compatibility condition

$$\overline{g}([\omega_i^\varphi] \cup [\omega_k^\varphi], [\omega_j^\varphi]) = \overline{g}([\omega_i^\varphi], [\omega_j^\varphi] \cup [\omega_k^\varphi])$$

but this follows from a straightforward computation of the right term and the left term, keeping in mind the definition of \overline{g} and \cup . The second follows from section 5: the isomorphism is induced by γ . □

Of course, this result should be compared with [22, Theorem 1.1].

Example 6.2.2. $w_0 = 1, w_1 = w_2 = 2$: the table of the orbifold cup-product \cup_{orb} is

\cup_{orb}	1	P	P^2	$\mathbf{1}_{\frac{1}{2}}$	$\mathbf{1}_{\frac{1}{2}}P$
1	1	P	P^2	$\mathbf{1}_{\frac{1}{2}}$	$\mathbf{1}_{\frac{1}{2}}P$
P		P^2	0	$\mathbf{1}_{\frac{1}{2}}P$	0
P^2			0	0	0
$\mathbf{1}_{\frac{1}{2}}$				P	P^2
$\mathbf{1}_{\frac{1}{2}}P$					0

and the one of \cup is

\cup	$[\omega_0^\varphi]$	$[\omega_1^\varphi]$	$[\omega_2^\varphi]$	$[\omega_3^\varphi]$	$[\omega_4^\varphi]$
$[\omega_0^\varphi]$	$[\omega_0^\varphi]$	$[\omega_1^\varphi]$	$[\omega_2^\varphi]$	$[\omega_3^\varphi]$	$[\omega_4^\varphi]$
$[\omega_1^\varphi]$		$[\omega_2^\varphi]$	0	$[\omega_4^\varphi]$	0
$[\omega_2^\varphi]$			0	0	0
$[\omega_3^\varphi]$				$\frac{1}{16}[\omega_1^\varphi]$	$\frac{1}{16}[\omega_2^\varphi]$
$[\omega_4^\varphi]$					0

7. CONSTRUCTION OF FROBENIUS MANIFOLDS

First, we recall how to construct Frobenius manifolds, starting from a Frobenius type structure (our references will be [11] and [18]): one needs a homogeneous and primitive section yielding an invertible period map. We then use this construction to define a limit Frobenius manifold, by unfolding the limit Frobenius type structure \mathbb{F}_w defined in section 6.1. Last, we end with a discussion about *logarithmic* Frobenius manifolds, as defined in [23].

7.1. Frobenius manifolds on $\mathcal{M} = \mathbb{C}^*$. Let Δ be an open disc in \mathcal{M} . The w -Frobenius type structure \mathbb{F}_w (see definition 5.1.2) gives also an analytic Frobenius type structure

$$\mathcal{F} = (\Delta, E^{an}, R_0^{an}, R_\infty, \Phi^{an}, \nabla^{an}, g^{an})$$

on the simply connected domain Δ . Universal deformations of this Frobenius type structure are defined in [11, Definition 2.3.1] and [18]. The following results are shown and discussed in detail in [11] in a slightly different situation, but the arguments in *loc. cit.* can be repeated almost verbatim here so we give only a skeetch of the proofs.

We keep in this section the notations of section 4. Let ω_0^{an} be the class of ω_0^φ in E^{an} : ω_0^{an} is ∇^{an} -flat because $R^\varphi(\omega_0^\varphi) = 0$.

Lemma 7.1.1. (1) *The Frobenius type structure \mathcal{F} has a universal deformation*

$$\tilde{\mathcal{F}} = (N, \tilde{E}^{an}, \tilde{R}_0^{an}, \tilde{R}_\infty, \tilde{\Phi}^{an}, \tilde{\nabla}^{an}, \tilde{g}^{an})$$

parametrized by $N := \Delta \times (\mathbb{C}^{\mu-1}, 0)$.

(2) *Let $\tilde{\omega}_0^{an}$ be the $\tilde{\nabla}^{an}$ -flat extension of ω_0^{an} . The period map*

$$\varphi_{\tilde{\omega}_0^{an}} : \Theta_N \rightarrow \tilde{E}^{an}$$

defined by $\varphi_{\tilde{\omega}_0^{an}}(\xi) = -\tilde{\Phi}_\xi^{an}(\tilde{\omega}_0^{an})$ is an isomorphism which makes N a Frobenius manifold.

Proof. (1) We can use the adaptation of [18, Theorem 2.5] given in [11, Section 6] because

$$\omega_0^{an}, R_0^{an}(\omega_0^{an}), \dots, (R_0^{an})^{\mu-1}(\omega_0^{an})$$

generate E^{an} and because $u_0 := 1/u_1^{w_1} \cdots u_n^{w_n}$ is not equal to zero in E^{an} . (2) follows from (1) (see e.g. [18, Theorem 4.5]). \square

The previous construction can be also done in the same way "point by point" (see [14] and [18] and the references therein) and this is the classical point of view: if $x \in \Delta$ one can attach to the Laurent polynomial $F_x := F(\cdot, x)$ a Frobenius type structure on a point \mathcal{F}_x^{pt} , a universal deformation $\tilde{\mathcal{F}}_x^{pt}$ of it, again because u_0 and its powers generate $\mathbb{C}[u, u^{-1}](\partial_{u_i} F_x)$, and finally a Frobenius structure on $M := (\mathbb{C}^\mu, 0)$ with the help of the section ω_0 . We will call it "the Frobenius structure attached to F_x ". Let \mathcal{F}_x (*resp.* $\tilde{\mathcal{F}}_x$) be the germ of \mathcal{F} (*resp.* $\tilde{\mathcal{F}}$) at $x \in \Delta$ (*resp.* $(x, 0)$).

Proposition 7.1.2. (1) *The deformations $\tilde{\mathcal{F}}_x$ and $\tilde{\mathcal{F}}_x^{pt}$ are isomorphic.*

(2) *The period map defined by the flat extension of ω_0^{an} to $\tilde{\mathcal{F}}_x$ is an isomorphism. This yields a Frobenius structure on M which is isomorphic to the one attached to F_x .*

Proof. Notice first that $\tilde{\mathcal{F}}_x^{pt}$ is a deformation of \mathcal{F}_x : this follows from the fact that u_0 does not belong to the Jacobian ideal of f : see [11, section 7]. Better, $\tilde{\mathcal{F}}_x^{pt}$ is a universal deformation of \mathcal{F}_x because \mathcal{F}_x is a deformation of \mathcal{F}_x^{pt} . This gives (1) because, by definition, two universal deformations of a same Frobenius type structure are isomorphic. (2) is then clear. \square

As a consequence, the universal deformations $\tilde{\mathcal{F}}_x^{pt}$, $x \in \Delta$, are the germs of a same section, namely $\tilde{\mathcal{F}}$. Thus, the Frobenius structure attached to F_{x_1} , $x_1 \in \Delta$, can be seen as an analytic continuation of the one attached to F_{x_0} , $x_0 \in \Delta$.

7.2. Limit Frobenius manifolds. We start from the canonical limit structures (see section 6.1.2) to construct limit Frobenius manifolds. We mime the process explained in section 7.1: the main point is to find an unfolding of our limit Frobenius type structure $\overline{\mathbb{F}}_w$ such that the associated period map is an isomorphism. In order to do this, we first unfold the Saito structure $\overline{\mathcal{S}}_w$ (which is after all a vector bundle with connection) and then we use proposition 2.2.5.

It should be emphasized that the cases $\mu = n + 1$ (manifold) and $\mu \geq n + 2$ (orbifold) will yield different conclusions.

7.2.1. Unfoldings of the canonical limit structures. The first step is thus to unfold the canonical limit Saito structure

$$\overline{\mathcal{S}}_w = (\overline{H}, \overline{\nabla}, \overline{S}, n)$$

(see definition 6.1.5). A basis of global sections of \overline{H} is $e = (e_0, \dots, e_{\mu-1})$ where we put $e_i := [\omega_i^\varphi]$ (remember that $[\omega_i^\varphi]$ denotes the class of ω_i^φ in \overline{H}). Recall the matrices \overline{A}_0 and A_∞ defined in section 6.1.

Define, for $i = 0, \dots, \mu - 1$, the matrices C_i by

$$C_i(e_j) = \begin{cases} -\frac{1}{w^w} e_{i+j} & \text{if } i + j \geq \mu \text{ and } 1 + c_{i+j} = c_i + c_j, \\ -e_{i+j} & \text{if } i + j \leq \mu - 1 \text{ and } c_{i+j} = c_i + c_j, \\ 0 & \text{otherwise} \end{cases}$$

and put

$$\tilde{A}_0(\underline{x}) = (\alpha_0 - 1)x_0 C_0 - \mu C_1 + (\alpha_2 - 1)x_2 C_2 + \dots + (\alpha_{\mu-1} - 1)x_{\mu-1} C_{\mu-1}$$

where $\underline{x} = (x_0, \dots, x_{\mu-1})$ is a system of coordinates on $M = (\mathbb{C}^\mu, 0)$ (with the previous notations, we have $x_1 = x$). Notice that $-\mu C_1 = \overline{A}_0$

Example 7.2.1. Assume that $w_1 = \dots = w_n = 1$. Then $\mu = n + 1$,

$$\overline{A}_0 = (n + 1) \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}$$

and $A_\infty = \text{diag}(0, 1, \dots, n)$. Put $J = \overline{A}_0 / (n + 1)$: we have $C_i = -J^i$ for $i = 0, \dots, n$, and

$$\tilde{A}_0(x) = -x_0 C_0 - (n + 1) C_1 + x_2 C_2 + 2x_3 C_3 + \dots + (n - 1) x_n C_n.$$

◆

Let \tilde{H} be the trivial bundle on $\mathbb{P}^1 \times M$ with basis $\tilde{e} = (\tilde{e}_0, \dots, \tilde{e}_{\mu-1}) = (1 \otimes e_0, \dots, 1 \otimes e_{\mu-1})$. Define on \tilde{H} the connection $\tilde{\nabla}$ whose matrix in the basis \tilde{e} is

$$\left(\frac{\tilde{A}_0(x)}{\theta} + A_\infty \right) \frac{d\theta}{\theta} + \theta^{-1} \sum_{i=0}^{\mu-1} C_i dx_i.$$

Define \tilde{S} on \tilde{H} by $\tilde{S}(\tilde{e}_i, \tilde{e}_j) = \overline{S}(e_i, e_j)$, this equality being extended by linearity.

Proposition 7.2.2. (1) *The tuple*

$$\tilde{\mathcal{S}}_w = (M, \tilde{H}, \tilde{\nabla}, \tilde{S}, n)$$

is a Saito structure which unfolds $\overline{\mathcal{S}}_w$.

(2) *Assume moreover that $w_0 = w_1 = \dots = w_n = 1$. Then the unfolding $\tilde{\mathcal{S}}_w$ is universal.*

Proof. (1) We have to show that $\tilde{\nabla}$ is flat and that \tilde{S} is $\tilde{\nabla}$ -flat. The flatness is equivalent to the equalities

$$\frac{\partial C_i}{\partial x_j} = \frac{\partial C_j}{\partial x_i}, [C_i, C_j] = 0$$

$$[\tilde{A}_0(x), C_i] = 0, \frac{\partial \tilde{A}_0}{\partial x_i} + C_i = [A_\infty, C_i]$$

for all i, j . Notice first that we have $C_i(e_0) = -e_i$ for $i = 0, \dots, \mu - 1$. We have

$$C_i C_j(e_k) = \begin{cases} e_{i+j+k} & \text{if } c_{i+j+k} = c_i + c_j + c_k, \\ e_{i+\overline{j+k}} & \text{if } 1 + c_{i+\overline{j+k}} = c_i + c_j + c_k, \\ e_{\overline{i+j+k}} & \text{if } 1 + c_{\overline{i+j+k}} = c_i + c_j + c_k, \\ e_{\overline{\overline{i+j+k}}} & \text{if } 2 + c_{\overline{\overline{i+j+k}}} = c_i + c_j + c_k \end{cases}$$

This is symmetric in i, j and thus $[C_i, C_j] = 0$. Now if we define

$$\tilde{A}_0(x) = \sum_{i=0}^{\mu-1} ([A_\infty, C_i] - C_i)x_i - \mu C_1$$

the conditions $\frac{\partial \tilde{A}_0}{\partial x_i} + C_i = [A_\infty, C_i]$ for all $i, j = 0, \dots, \mu - 1$ are obviously satisfied. But we have also $[A_\infty, C_i] = \alpha_i C_i$, because the condition $1 + c_{i+\overline{j}} = c_i + c_j$ (*resp.* $c_{i+j} = c_i + c_j$) is equivalent to $\alpha_{i+\overline{j}} = \alpha_i + \alpha_j$ (*resp.* $\alpha_{i+j} = \alpha_i + \alpha_j$), hence $[\tilde{A}_0(x), C_i] = 0$ and the connection is flat. For the $\tilde{\nabla}$ -flatness of \tilde{S} , it is enough to notice that $C_i^* = C_i$, $*$ denoting the adjoint with respect to $\overline{\mathcal{S}}_w$. This is shown using the kind of computations above. For the second assertion, notice that $\tilde{A}_0(0) = \overline{A}_0$.

(2) If $w_0 = \dots = w_n = 1$, e_0 induces a cyclic vector of \overline{A}_0 . Hence, we can use [18, p. 123]: the universality then follows from the fact that $(C_i)_{i+1,1} = -1$ for all $i = 0, \dots, \mu - 1$. \square

The Saito structure $\tilde{\mathcal{S}}_w$, with the help of proposition 2.2.5, gives a Frobenius type structure on M ,

$$\tilde{\mathbb{F}}_w = (M, \tilde{E}, \tilde{\nabla}, \tilde{R}_0, \tilde{R}_\infty, \tilde{\Phi}, \tilde{g})$$

the matrices of \tilde{R}_0 and \tilde{R}_∞ being, in the obvious bases, \tilde{A}_0 and $-A_\infty$. By definition, it is an unfolding of $\overline{\mathbb{F}}_w$.

7.2.2. Construction of limit Frobenius manifolds. In order to get a Frobenius manifold from Frobenius type structure $\tilde{\mathbb{F}}_w$, we still need an invertible period map: its existence follows from the choice of the first columns of the matrices C_i .

Corollary 7.2.3. (1) *The period map*

$$\varphi_{\tilde{e}_0} : TM \rightarrow \tilde{E},$$

defined by $\varphi_{\tilde{e}_0}(\xi) = -\tilde{\Phi}_\xi(\tilde{e}_0)$, is an isomorphism and \tilde{e}_0 is an eigenvector of \tilde{R}_∞ .

(2) *The section \tilde{e}_0 defines, through the period map $\varphi_{\tilde{e}_0}$ a Frobenius structure on M which makes M the canonical limit Frobenius manifold for which:*

- (a) *the coordinates $(x_0, \dots, x_{\mu-1})$ are ∇ -flat: one has $\nabla \partial_{x_i} = 0$ for all $i = 0, \dots, \mu - 1$,*
- (b) *the product is constant in flat coordinates,*
- (c) *the potential Ψ is a polynomial of degree less or equal to 3,*

(d) the Euler vector field is $E = -(\alpha_0 - 1)x_0\partial_{x_0} + \mu\partial_{x_1} - (\alpha_2 - 1)x_2\partial_{x_2} - \cdots - (\alpha_{\mu-1} - 1)x_{\mu-1}\partial_{x_{\mu-1}}$.

Proof. (1) Indeed, the period map $\varphi_{\tilde{e}_0}$ is defined by $\varphi_{\tilde{e}_0}(\partial_{x_i}) = -C_i(\tilde{e}_0) = \tilde{e}_{i-1}$. Last, \tilde{e}_0 is an eigenvector of \tilde{R}_∞ because e_0 is an eigenvector of R_∞ . Let us show (2): the isomorphism $\varphi_{\tilde{e}_0}$ brings on TM the structures on \tilde{E} : (a) follows from the fact that the first column of the matrices C_i are constant and (b) from the fact that the matrices C_i are constant because, by the definition of the product, $\varphi_{\tilde{e}_0}(\partial_{x_i} * \partial_{x_j}) = C_i(C_j(\tilde{e}_0))$; (c) follows from (b) because, in flat coordinates,

$$g(\partial_{x_i} * \partial_{x_j}, \partial_{x_k}) = \frac{\partial^3 \Psi}{\partial x_i \partial x_j \partial x_k}$$

where g is the metric on TM induced by \tilde{g} . Last, (d) follows from the definition of $\tilde{A}_0(x)$. \square

Remark 7.2.4. If $w_1 = \cdots = w_n = 1$, the product is given by $\partial_{x_i} * \partial_{x_j} = \partial_{x_{i+j}}$ if $i+j \leq \mu-1$, 0 otherwise, and we have

$$\Psi = \sum_{i,j, i+j \leq \mu-1} \frac{1}{6} x_i x_j x_{\mu-1-i-j}$$

up to a polynomial of degree less or equal to 2. \blacklozenge

Remark 7.2.5. Of course, the period map can be an isomorphism for other choices of the first columns of the matrices C_i :

- the resulting Frobenius manifolds will be isomorphic to the one given by the corollary if $w_1 = \cdots = w_n = 1$ (manifold case) because the Frobenius type structure $\tilde{\mathbb{F}}_w$ is a *universal* deformation of our limit Frobenius type structure $\overline{\mathbb{F}}_w$ (see [18] and [11, Theorem 3.2.1]). We will thus call the Frobenius manifold described above the *canonical limit Frobenius manifold*. This Frobenius structure is the one on $M := H^*(\mathbb{P}^n, \mathbb{C})$ given by the cup product and the Poincaré duality on each tangent spaces.
- If there exists an w_i such that $w_i \geq 2$ (orbifold case), one could get, starting from $\overline{\mathbb{F}}_w$, several Frobenius manifolds (we have shown that there exists at least one), which can be difficult to compare: we loose the universality property here. However, the Frobenius manifold obtained in the previous corollary is the one on $M := H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$ given by the orbifold cup product and the Poincaré duality on each tangent spaces.

\blacklozenge

7.3. Logarithmic Frobenius manifolds. A manifold M is a *Frobenius manifold with logarithmic poles along the divisor* $D = \{x = 0\}$ (for short a *logarithmic Frobenius manifold*) if $\text{Der}_M(\log D)$ is equipped with a metric, a multiplication and two (global) logarithmic vector fields (the unit e for the multiplication and the Euler vector field E), all these objects satisfying the usual compatibility relations (see [23, Definition 1.4]). We can also define a Frobenius manifold with logarithmic poles *without metric*: in this case, we still need a flat, torsionless connection, a symmetric Higgs field (that is a product) and two global logarithmic vector fields as before.

There are two ways to construct such manifolds: the first one is to start from initial data, namely a logarithmic Frobenius type structure in the sense of [23, Definition 1.6], and to unfold it, just as in section 7.1. This logarithmic Frobenius type structure will be obtained from a logarithmic Saito structure, as in proposition 2.2.5. The second is to work directly with the big Gromov-Witten potential, as it is done in *loc. cit.* in the case of \mathbb{P}^n . We explore these two ways.

7.3.1. *Construction via unfoldings.* Let $N = \mathbb{C}$. We will denote the coordinate on N by x and we will put $D := \{x = 0\}$. The following definitions are borrowed from [23].

Definition 7.3.1. A Saito structure of weight n on $\mathbb{P}^1 \times N$ with *logarithmic poles along D* (for short a *logarithmic Saito structure*) is a tuple

$$(N, D, H^{log}, \nabla^{log}, S^{log}, n)$$

where H^{log} is a trivial bundle on $\mathbb{P}^1 \times N$, ∇^{log} is a flat meromorphic connection on H^{log} such that

$$\nabla^{log}(\Gamma(\mathbb{P}^1 \times N, H^{log})) \subset \theta^{-1}\Omega_{\mathbb{C} \times N}^1(\log((\{0\} \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}))) \otimes \Gamma(\mathbb{P}^1 \times N, H^{log})$$

and S^{log} is a ∇^{log} -flat bilinear form as in definition 2.2.1.

In order to construct logarithmic Frobenius manifolds, we will need the following

Definition 7.3.2. A Frobenius type structure with *logarithmic pole along D* (for short, a *logarithmic Frobenius type structure*) is a tuple

$$(N, D, E^{log}, \nabla^{log}, R_0^{log}, R_\infty^{log}, \Phi^{log}, g^{log})$$

where E^{log} is a bundle on N , R_0^{log} and R_∞^{log} are \mathcal{O}_N -linear endomorphisms of E^{log} ,

$$\Phi^{log} : E^{log} \rightarrow \Omega^1(\log(D)) \otimes E^{log}$$

is a \mathcal{O}_N -linear map, g^{log} is a metric on E^{log} , i.e a \mathcal{O}_N -bilinear form, symmetric and non-degenerate, and ∇^{log} is a connection on E^{log} with logarithmic pole along D , these object satisfying the compatibility relations of section 2.2.

Remark 7.3.3. (1) One can also define in an obvious way a logarithmic Saito structures and logarithmic Frobenius type structure *without metric*.

(2) As in section 2.2, a logarithmic Saito structure determines a logarithmic Frobenius type structure (see [23, proposition 1.10])

(3) As before, we will work preferably in the algebraic category: E^{log} will be a free $\mathbb{C}[x]$ -module *etc...* ♦

Proposition 3.3.7 and theorem 4.3.2 suggests that we are not so far from a logarithmic Saito structure. Indeed, with the notations of section 4 and forgetting the index B , H^{log} will be obtained from an extension of G_0 as a free $\mathbb{C}[x, \theta]$ -module (recall that G_0 is only a $\mathbb{C}[x, x^{-1}, \theta]$ -module). We can use for instance the $\mathbb{C}[x, \theta]$ -submodule of G_0 generated by $\omega_0^\varphi, \dots, \omega_{\mu-1}^\varphi$, and we thank C. Sevenheck for this suggestion: we will denote it by \mathcal{L}_0^φ . Let $\mathcal{L}_\infty^\varphi$ be the $\mathbb{C}[x, \tau]$ -module generated by $\omega_0^\varphi, \dots, \omega_{\mu-1}^\varphi$ where, as usual, $\tau := \theta^{-1}$. These two free modules give a trivial bundle H^{log} equipped with a connection with the desired poles, thanks to theorem 4.3.2. In order to define the metric S^{log} , extend the bilinear form S defined in section 4.4 to \mathcal{L}_0^φ . We will denote the resulting tuple by \mathcal{S}_w^{log} .

The logarithmic Frobenius type structure is then obtained as follows: put $E^{log} = \mathcal{L}_0^\varphi / \theta \mathcal{L}_0^\varphi$. Define, as in section 2.2, the endomorphisms R_0^{log} and Φ_ξ^{log} for any logarithmic vector field $\xi \in Der_{\mathbb{C}}(\log D)$ and, using now the restriction of $\mathcal{L}_\infty^\varphi$ at $\tau = 0$, the endomorphisms R_∞^{log} and ∇_ξ^{log} . We get the flat bilinear symmetric form g^{log} on E^{log} putting

$$g^{log}([\omega_i^\varphi], [\omega_j^\varphi]) := \theta^{-n} S^{log}(\omega_i^\varphi, \omega_j^\varphi)$$

where $[]$ denotes the class in E^{log} . We will denote the resulting tuple by \mathbb{F}_w^{log} .

Proposition 7.3.4. (1) The tuple \mathcal{S}_w^{log} is a logarithmic Saito structure if $w_0 = \dots = w_n = 1$ and a logarithmic Saito structure without metric otherwise.

(2) The tuple \mathbb{F}_w^{log} is a logarithmic Frobenius type structure if $w_0 = \dots = w_n = 1$ and a logarithmic Frobenius type structure without metric otherwise.

Proof. By section 4.4, S^{\log} is not nondegenerate, unless $w_0 = \dots = w_n = 1$. This gives (1) and (2) follows. \square

Corollary 7.3.5. *The section ω_0^φ together with the tuple \mathbb{F}_w^{\log} define a logarithmic Frobenius manifold if $w_0 = \dots = w_n = 1$ and a logarithmic Frobenius manifold without metric otherwise.*

Proof. Define

$$\varphi_{\omega_0^\varphi} : \text{Der}_{\mathbb{C}}(\log D) \rightarrow E^{\log},$$

by $\varphi_{\omega_0^\varphi}(\xi) := -\Phi_\xi^{\log}(\omega_0^\varphi)$. By theorem 4.3.2, the matrix of $\Phi_{x\partial_x}^{\log}$ is $-A_0^\varphi(x)\mu^{-1}$. Hence $\varphi_{\omega_0^\varphi}|_0$ is injective and $\omega_0^\varphi|_0$ and its images under iteration of the maps $\Phi_{x\partial_x}^{\log}|_0$ generate $E^{\log}|_0$. The result now follows from [23, theorem 1.12] because the section ω_0^φ satisfies conditions (IC), (EC) and (GC) of *loc. cit.* and its restriction to $N - D$ is ∇^{\log} -flat (because $R_\infty^{\log}(\omega_0^\varphi) = 0$). \square

If $w_0 = \dots = w_n = 1$, we thus get a counterpart of the results obtained for \mathbb{P}^n , by a different method (see section below) in [23, section 2]. If there exists a weight w_i such that $w_i \geq 2$, the construction of a logarithmic Frobenius manifold with metric using this method is still an open problem.

Remark 7.3.6. One could of course consider different extensions of G_0 as a free $\mathbb{C}[x, \theta]$ -module and start with a different logarithmic Saito structure: for instance, it is possible to work with the lattice \mathcal{L}_0^ψ such that the eigenvalues of the residue matrix of ∇_{∂_x} at $x = 0$ are contained in $] -1, 0]$. It is easily checked that (with obvious notations) the section ω_0^ψ in \mathcal{L}_0^ψ is flat but does not satisfy (GC) if $\mu \geq n + 2$. The only section which satisfies (IC), (EC) and (GC) is ω_{n+1}^ψ but this one is not flat. \blacklozenge

7.3.2. Construction via the Gromov-Witten potential. In [23], Reichelt associates a logarithmic Frobenius manifold to a smooth projective variety, using the Gromov-Witten potential. In this section, we explain why his construction does not apply in the orbifold case.

In order to simplify the notations, we focus on weighted projective spaces. Put $M_A := H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$ and let $(M_A, H^A, \nabla^A, S^A, n)$ be its big A -model D -module (see Definition 3.1.5). We define the action of $\text{Pic}(\mathbb{P}(w))$ on the trivial bundle $H^A \rightarrow \mathbb{P}^1 \times M_A$ as follows:

- (1) on the fibers of H^A we define, for any $f \in F$ and $\alpha_f \in H^*(\mathbb{P}(w)_{S_f}, \mathbb{C})$,

$$\mathcal{O}(d) \cdot \alpha_f := e^{2\pi\sqrt{-1}df} \alpha_f$$

- (2) on $M_A = H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$ we define

$$\mathcal{O}(d) \cdot \left(\alpha \oplus \bigoplus_{f \in F/\{0\}} \alpha_f \right) := (\alpha - 2\pi\sqrt{-1}d \cdot c_1(\mathcal{O}(1))) \oplus \bigoplus_{f \in F/\{0\}} e^{2\pi\sqrt{-1}d \cdot f} \alpha_f$$

As in proposition 3.2.2, the Saito structure is equivariant with respect to this action so that we have a quotient Saito structure $(\mathcal{M}_A, \tilde{H}^A, \tilde{\nabla}^A, \tilde{S}^A, n)$ where $\mathcal{M}_A := M_A / \text{Pic}(\mathbb{P}(w))$. As the basis $(\mathbf{1}_f P^k)$ is not invariant for $f \neq 0$ with respect to this action on M_A (see Proposition 3.3.3), the associated coordinates $(t_0, q = e^{t_1}, t_2, \dots, t_{\mu-1})$ on M_A are not coordinates on the quotient \mathcal{M}_A . Nevertheless, we can complete $(t_0, q = e^{t_1}, t_2, \dots, t_n)$ in order to get a system of coordinates, denoted by $\underline{t} = (t_0, q = e^{t_1}, t_2, \dots, t_n, \tau_{n+1}, \dots, \tau_{\mu-1})$, on \mathcal{M}_A .

Put $\tilde{E}^A := \tilde{H}^A|_{\{0\} \times \mathcal{M}_A}$. If we want to repeat the argument given by Reichelt in §2.1.1 [23], we should define the metric using a “infinitesimal period map” $T\mathcal{M}_A \rightarrow \tilde{E}^A$ which sends the vector field ∂_{τ_i} to $\mathbf{1}_{c_i} P^{r(i)}$ (cf (30) for the notation). This is not allowed in the orbifold case because for $c_i \neq 0$ the cohomology class $\mathbf{1}_{c_i} P^{r(i)}$ does not define a global section of the quotient bundle $\tilde{H}^A \rightarrow \mathbb{P}^1 \times \mathcal{M}_A$.

Natural global sections of \widetilde{E}^A are $(P^{\bullet\tau^i})_{i \in \{0, \dots, \mu-1\}}$. But proposition 3.3.5 implies that the metric degenerates at $q = 0$. Hence as in corollary 7.3.5, using these global sections, we get a logarithmic Frobenius manifold without metric on \mathcal{M}_A in the orbifold case.

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